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DERIVATION OF TWO CONTIGUOUS FORMULAS OF KUMMER'S SECOND THEOREM VIA DIFFERENTIAL EQUATION

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Abstract

We aim to provide an alternative proof of two transformation formulas contiguous to Kummer's second transformation for the confluent hypergeometric function ${}_1F_1$ using a differential equation approach.

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1. Introduction and Preliminaries

We start with the Kummer's second transformation [2] for the confluent hypergeometric function given by

$$e^{-z}\,{}_1F_1\left[\begin{matrix} a; \\ 2a; \end{matrix}\ 2z\right] = {}_0F_1\left[\begin{matrix} -; \\ a+\frac{1}{2}; \end{matrix}\ \frac{z^2}{4}\right], \tag{1.1}$$

which is valid when $2a$ is neither zero nor negative integer. The transformation (1.1) was derived with the aid of the differential equation satisfied by ${}_1F_1$ (see, e.g., [6, p. 126]). Bailey [1] re-derived this result by employing the Gauss second summation theorem (see, e.g., [5]):

$${}_2F_1\left[\begin{matrix} a,\ b, \\ \frac{1}{2}(a+b+1); \end{matrix}\ \frac{1}{2}\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)} \tag{1.2}$$

and Rathie and Choi [7] obtained the result by using the classical Gauss summation theorem (see, e.g., [6]):

$${}_2F_1\left[\begin{matrix} a,\ b; \\ c; \end{matrix}\ 1\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\ (\Re(c-a-b)>0). \tag{1.3}$$

Rathie and Nagar [8] established two transformation formulas contiguous to (1.1) with the help of contiguous forms of Gauss's second summation theorem (see [5]), which are recalled in the following theorem. Throughout this paper, let $\mathbb{C}$, $\mathbb{Z}$ and $\mathbb{N}$ be the sets of complex numbers, integers and positive integers, respectively, and $\mathbb{Z}_0^- := \mathbb{Z}\backslash\mathbb{N}$ and $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$.

Theorem 1.1. *For* $2a\pm 1 \in \mathbb{C}\backslash\mathbb{Z}_0^-$, *the following formulas hold true*:

$$e^{-z}\,{}_1F_1\left[\begin{matrix} a; \\ 2a+1; \end{matrix}\ 2z\right] = {}_0F_1\left[\begin{matrix} -; \\ a+\frac{1}{2}; \end{matrix}\ \frac{z^2}{4}\right] - \frac{z}{2a+1}\,{}_0F_1\left[\begin{matrix} -; \\ a+\frac{3}{2}; \end{matrix}\ \frac{z^2}{4}\right] \tag{1.4}$$

and

$$e^{-z}\,{}_1F_1\left[\begin{matrix}a;\\ 2a-1;\end{matrix}\ 2z\right]={}_0F_1\left[\begin{matrix}-;\\ a-\frac{1}{2};\end{matrix}\ \frac{z^2}{4}\right]+\frac{z}{2a-1}\,{}_0F_1\left[\begin{matrix}-;\\ a+\frac{1}{2};\end{matrix}\ \frac{z^2}{4}\right]. \quad (1.5)$$

Kim et al. [3] generalized the Kummer's second theorem (1.1) to give explicit expressions of the following forms:

$$e^{-z}\,{}_1F_1\left[\begin{matrix}a;\\ 2a+i;\end{matrix}\ 2z\right] \quad (i=0,\ \pm1,\ \pm2,\ \pm3). \quad (1.6)$$

Here we are interested in the results for $i=\pm2$ which are given in the following theorem:

Theorem 1.2. *For* $2a\pm2\in\mathbb{C}\backslash\mathbb{Z}_0^-$, *the following formulas hold true*:

$$e^{-z}\,{}_1F_1\left[\begin{matrix}a;\\ 2a+2;\end{matrix}\ 2z\right]={}_0F_1\left[\begin{matrix}-;\\ a+\frac{3}{2};\end{matrix}\ \frac{z^2}{4}\right]-\frac{z}{a+1}\,{}_0F_1\left[\begin{matrix}-;\\ a+\frac{3}{2};\end{matrix}\ \frac{z^2}{4}\right]$$
$$+\frac{z^2}{(a+1)(2a+3)}\,{}_0F_1\left[\begin{matrix}-;\\ a+\frac{5}{2};\end{matrix}\ \frac{z^2}{4}\right] \quad (1.7)$$

and

$$e^{-z}\,{}_1F_1\left[\begin{matrix}a;\\ 2a-2;\end{matrix}\ 2z\right]={}_0F_1\left[\begin{matrix}-;\\ a-\frac{1}{2};\end{matrix}\ \frac{z^2}{4}\right]+\frac{z}{a-1}\,{}_0F_1\left[\begin{matrix}-;\\ a-\frac{1}{2};\end{matrix}\ \frac{z^2}{4}\right]$$
$$+\frac{z^2}{(a-1)(2a-1)}\,{}_0F_1\left[\begin{matrix}-;\\ a+\frac{1}{2};\end{matrix}\ \frac{z^2}{4}\right]. \quad (1.8)$$

Very recently, Kodavanji et al. [2] established the results (1.4) and (1.5) by using the involved differential equations.

The objective of this note is to give an alternative proof of the contiguous transformations (1.7) and (1.8) by using the differential equation approach (see, e.g., [6, p. 126]). It is worth remarking that these transformations (1.7) and (1.8) cannot be derived by using only the hypergeometric differential

equation but can be proved by using a related second order differential equation with the aid of the standard Frobenius method.

2. Derivation of (1.1) by Rainville's Method

Here we present an outline of the arguments employed by Rainville [6, p. 126] who used to establish the Kummer transformation (1.1).

The confluent hypergeometric function ${}_1F_1\left[\begin{matrix}a;\\ b;\end{matrix}\ x\right]$ satisfies the following differential equation (see [4, Equation (13.2.1)]):

$$x\frac{d^2w}{dx^2}+(b-x)\frac{dw}{dx}-aw=0. \tag{2.1}$$

If we put $b=2a,$ make the change of variable x into $2z,$ and let $w=e^z y,$ then (2.1) becomes

$$z\frac{d^2y}{dz^2}+2a\frac{dy}{dz}-zy=0, \tag{2.2}$$

one of whose solutions is

$$y=e^{-z}\,{}_1F_1\left[\begin{matrix}a;\\ 2a;\end{matrix}\ 2z\right] \quad (2a\in\mathbb{C}\backslash\mathbb{Z}_0^-). \tag{2.3}$$

The differential equation (2.2) is invariant under the change of variable from z to $-z.$ Therefore, if we introduce a new independent variable $\sigma=\frac{z^2}{4},$ then the equation with the variable y becomes

$$\sigma^2\frac{d^2y}{d\sigma^2}+\left(a+\frac{1}{2}\right)\sigma\frac{dy}{d\sigma}-\sigma y=\sigma\frac{d}{d\sigma}\left(\sigma\frac{d}{d\sigma}+a-\frac{1}{2}-\sigma\right)y=0, \tag{2.4}$$

which is the differential equation for the ${}_0F_1$ function. Two linearly independent solutions are given by (see [4, 16.8(ii)])

$${}_0F_1\left[\begin{matrix}-;\\ a+\frac{1}{2};\end{matrix}\ \sigma\right] \quad \text{and} \quad \sigma^{\frac{1}{2}-a}\,{}_0F_1\left[\begin{matrix}-;\\ \frac{3}{2}-a;\end{matrix}\ \sigma\right].$$

We then find that, if $a+\frac{1}{2}$ is non-integer (that is, if $2a$ is not an odd integer),

$$y = A\,{}_0F_1\left[\begin{matrix}-;\\ a+\frac{1}{2};\end{matrix}\ \frac{z^2}{4}\right] + Bz^{1-2a}\,{}_0F_1\left[\begin{matrix}-;\\ \frac{3}{2}-a;\end{matrix}\ \frac{z^2}{4}\right], \tag{2.5}$$

where A and B are arbitrary constants. Since (2.3) is also a solution of the differential equation (2.4), we have

$$e^{-z}\,{}_1F_1\left[\begin{matrix}a;\\ 2a;\end{matrix}\ 2z\right] = A\,{}_0F_1\left[\begin{matrix}-;\\ a+\frac{1}{2};\end{matrix}\ \frac{z^2}{4}\right] + Bz^{1-2a}\,{}_0F_1\left[\begin{matrix}-;\\ \frac{3}{2}-a;\end{matrix}\ \frac{z^2}{4}\right]. \tag{2.6}$$

The left-hand side and the first member on the right-hand side of the expression (2.6) are both analytic at $z = 0$ while the remaining term is not analytic at $z = 0$ due to the presence of the factor z^{1-2a}. Hence, $B = 0$ and setting $z = 0$ in the resulting equation is easily seen to give $A = 1$. When $2a$ is an odd positive integer, the second solution in (2.4) involves a log z term, and the same argument shows that $A = 1$ and $B = 0$. Hence, this proves the transformation (1.1).

3. An Alternative Derivation of Theorem 1.2

We first prove the contiguous transformation (1.7). Setting $b = 2a + 2$ in (2.1), changing the variable x into $2z$, and letting $w = e^z y$, we have

$$z\frac{d^2y}{dz^2} + (2a+2)\frac{dy}{dz} + (1-z)y = 0, \tag{3.1}$$

one of whose solutions is found to be

$$y = e^{-z}\,{}_1F_1\left[\begin{matrix}a;\\ 2a+2;\end{matrix}\ 2z\right] \quad (2a+2 \in \mathbb{C}\backslash\mathbb{Z}_0^-). \tag{3.2}$$

The differential equation (3.1) is not invariant when the variable z is changed into $-z$. So we cannot reduce the differential equation (3.1) to a differential equation for ${}_0F_1$. Yet an inspection of (3.1) shows that the point $z = 0$ is a regular singular point. Accordingly, we seek two linearly independent solutions of (3.1) by the Frobenius method. To do this, let

$$y = z^{\lambda}\sum_{n=0}^{\infty} c_n z^n \quad (c_0 \neq 0), \tag{3.3}$$

where λ is the indicial exponent. Substituting this y in (3.3) for the y in (3.1), after a little simplification, we obtain

$$\sum_{n=0}^{\infty} c_n(n+\lambda)(n+\lambda+2a+1)z^{n-1} + 2\sum_{n=0}^{\infty} c_n z^n = \sum_{n=0}^{\infty} c_n z^{n+1}. \tag{3.4}$$

The coefficient of z^{-1} in (3.4) is seen to vanish to yield the indicial equation:

$$\lambda(\lambda+1+2a) = 0, \tag{3.5}$$

from which we have $\lambda = 0$ or $\lambda = -1-2a$. Equating coefficients of z^n $(n \in \mathbb{N}_0)$, we obtain

$$c_1 = \frac{-2c_0}{(1+\lambda)(2+\lambda+2a)} \quad \text{and} \quad c_n = \frac{c_{n-2} - c_{n-1}}{(n+\lambda)(n+\lambda+2a+1)}$$

$$(n \in \mathbb{N}\backslash\{1\}). \tag{3.6}$$

Choosing $\lambda = 0$ in (3.6) gives

$$c_1 = \frac{-2c_0}{2a+2} \quad \text{and} \quad c_n = \frac{c_{n-2} - c_{n-1}}{n(n+2a+1)} \quad (n \in \mathbb{N}\backslash\{1\}). \tag{3.7}$$

Solution of this three-term recurrence (3.7), with the help of Mathematica, generates the following values: For $n \in \mathbb{N}_0$,

$$c_{2n} = \frac{2^{-2n}\left(\frac{a}{2}+\frac{3}{2}\right)_n c_0}{n!\left(\frac{a}{2}+\frac{1}{2}\right)_n\left(a+\frac{3}{2}\right)_n} \quad \text{and} \quad c_{2n+1} = \frac{2^{-2n}c_1}{n!\left(a+\frac{3}{2}\right)_n}, \tag{3.8}$$

whose general values were determined by induction. Replacing c_n in (3.3) by the c_n in (3.8) yields one solution of the differential equation (3.1) as follows:

$$y_1 = c_0\left\{{}_1F_2\left[\begin{matrix}\frac{a}{2}+\frac{3}{2};\\ \frac{a}{2}+\frac{1}{2},\ a+\frac{3}{2};\end{matrix}\ \frac{z^2}{4}\right]-\frac{z}{a+1}\,{}_0F_1\left[\begin{matrix}-;\\ a+\frac{3}{2};\end{matrix}\ \frac{z^2}{4}\right]\right\}. \tag{3.9}$$

The ${}_1F_2$ in (3.9) can be split into two ${}_0F_1$ as follows:

$$y_1 = c_0\left\{{}_0F_1\left[\begin{matrix}-;\\ a+\frac{3}{2};\end{matrix}\ \frac{z^2}{4}\right]-\frac{z}{a+1}\,{}_0F_1\left[\begin{matrix}-;\\ a+\frac{3}{2};\end{matrix}\ \frac{z^2}{4}\right]\right.$$

$$\left.+\frac{z^2}{(a+1)(2a+3)}\,{}_0F_1\left[\begin{matrix}-;\\ a+\frac{5}{2};\end{matrix}\ \frac{z^2}{4}\right]\right\}. \tag{3.10}$$

Taking $\lambda = -1-2a$ in (3.6) yields

$$c_1 = \frac{c_0}{a} \quad \text{and} \quad c_n = \frac{c_{n-2}-c_{n-1}}{n(n-2a-1)} \quad (n \in \mathbb{N}\backslash\{1\}),$$

from which, similarly as above, we obtain

$$c_{2n} = \frac{2^{-2n}\left(\frac{a}{2}-1\right)_n c_0}{n!\left(\frac{a}{2}\right)_n\left(\frac{1}{2}-a\right)_n} \quad \text{and} \quad c_{2n+1} = \frac{2^{-2n}c_1}{n!\left(\frac{1}{2}-a\right)_n}$$

$$(n \in \mathbb{N}_0;\ 2a-2 \in \mathbb{C}\backslash\mathbb{Z}). \tag{3.11}$$

Using the c_n in (3.11) for the c_n in (3.3), we obtain another solution of (3.1) as follows:

$$y_2 = c_0 z^{-1-2a}\left\{{}_1F_2\left[\begin{matrix}\frac{a}{2}-1;\\ \frac{a}{2},\ \frac{1}{2}-a;\end{matrix}\ \frac{z^2}{4}\right]+\frac{z}{a}\,{}_0F_1\left[\begin{matrix}-;\\ \frac{1}{2}-a;\end{matrix}\ \frac{z^2}{4}\right]\right\}. \tag{3.12}$$

The ${}_1F_2$ in (3.12) can be split into two ${}_0F_1$ as follows:

$$y_2 = c_0 z^{1-2a}\left\{ {}_0F_1\left[\begin{matrix} -; \\ \frac{1}{2}-a; \end{matrix}\ \frac{z^2}{4}\right] + \frac{z}{a}\,{}_0F_1\left[\begin{matrix} -; \\ \frac{1}{2}-a; \end{matrix}\ \frac{z^2}{4}\right] \right.$$
$$\left. + \frac{z^2}{a(1-2a)}\,{}_0F_1\left[\begin{matrix} -; \\ \frac{3}{2}-a; \end{matrix}\ \frac{z^2}{4}\right]\right\}. \tag{3.13}$$

A similar argument in Section 2 gives

$$e^{-z}\,{}_1F_1\left[\begin{matrix} a; \\ 2a+2; \end{matrix}\ 2z\right] = Ay_1 + By_2, \tag{3.14}$$

where A and B are constants. It is observed that the left-hand side of (3.14) and the y_1 are analytic at $z = 0$ while the y_2 is not analytic at $z = 0$ due to the presence of the factor z^{-2a-l}. Therefore, $B = 0$ and setting $z = 0$ in the resulting equation yields $A = 1$. When $2a \in \mathbb{N}$, the indicial exponents differ by an integer and y_2 involves a term in log z. So we again have $A = 1$ and $B = 0$. This proves the formula (1.7).

Similarly as above, we can also prove the formula (1.8). The detailed account of its proof is omitted.

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