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ON THE STRUCTURE OF FINITELY GENERATED PRIMARY MODULES

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Abstract

Given a finitely generated primary module over a principal ideal domain, we present the structure of its endomorphism ring and identify a fully invariant submodule in terms of a cyclic decomposition of the module. Furthermore, we use this identification to characterize S-prime submodules.

1. Introduction

Let R be a principal ideal domain, M be a finitely generated primary R-module, and $E = End_R(M)$. To begin with, we review the structure of the endomorphism ring $End_R(M)$ of the module M. Suppose

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$$M = \langle u_1 \rangle \oplus \cdots \oplus \langle u_m \rangle \quad o(u_i) = p^{e_i}, \quad e = e_1 \geq \cdots \geq e_m \geq 1, \tag{1}$$

where $o(u)$ denotes the order of u, is a cyclic decomposition of M (see [4]). Then, it is a routine to obtain a description of the endomorphism ring $End_R(M)$, as an R-module

$$End_R(M) = \oplus_{i=1}^{m} \oplus_{j=1}^{m} \langle \delta_{ij} \rangle, \tag{2}$$

where $\delta_{ij} \in End_R(M)$ is defined as follows: for every $1 \leq \mu \leq m$,

$$\delta_{ij}(u_\mu) = \begin{cases} p^{\max\{0, e_i - e_j\}} u_i, & \text{if } \mu = j, \\ 0, & \text{if } \mu \neq j. \end{cases} \tag{3}$$

According to above definition it is clear that $o(\delta_{ij}) = p^{\min\{e_i, e_j\}}$, and $\delta_{ij_o}\delta_{k_o j} = 0$ if $j_o \neq k_o$. For example, if $\delta_{32}, \delta_{22} \in End_R(M)$ and $b \in End_R(M)$, $b = \sum_{i=1,\, j=1}^{m} b_{ij}\delta_{ij}$, where $b_{ij} \in R$, then

$$\delta_{32} \cdot b \cdot \delta_{22}(u_2) = \delta_{32} \sum_{i=1}^{m} b_{i2}\delta_{i2}(u_2) = \delta_{32}\left(b_{22}\delta_{22}(u_2) + \sum_{i=1,\, i\neq 2}^{m} b_{i2}\delta_{i2}(u_2) \right)$$

$$= b_{22} \cdot \delta_{32} \cdot \delta_{22}(u_2) + 0 = b_{22} \cdot u_3 = b_{22}\delta_{32}(u_2).$$

Hence $\delta_{32} \cdot b \cdot \delta_{22} = b_{22}\delta_{32}$.

Furthermore, we have an important property, that is,

$$\delta_{ik_o}\delta_{k_o j} = \begin{cases} p^{e_{k_o} - e_j}\delta_{ij} & \text{if } k_o < j \quad \text{and } j < i, \\ p^{e_{k_o} - e_i}\delta_{ij} & \text{if } k_o < i \quad \text{and } i < j, \\ p^{e_j - e_{k_o}}\delta_{ij} & \text{if } k_o > j \quad \text{and } j > i, \\ p^{e_i - e_{k_o}}\delta_{ij} & \text{if } k_o > i \quad \text{and } i > j, \\ \delta_{ij} & \text{if } i \leq k_o \leq j \quad \text{or } j \leq k_o \leq i. \end{cases} \tag{4}$$

An R-submodule $X \subseteq M$ is a fully invariant submodule of M if and only if X is an E-submodule of M. Meanwhile, the class of primary modules has an important role in the study of modules over a principal ideal domain. Furthermore, we can see that a primary module is a self generator, and according to Theorem 1.10 in [2], $(X : M) = \{f \in E \mid f(M) \subseteq X\}$ is a prime ideal of E if and only if X is an S-prime submodule of M. Then how about the structure of the ideals of its endomorphism and a fully invariant submodule in terms of a cyclic decomposition of the module. In this paper, we investigate the structure of the endomorphism ring and identify the fully invariant submodule in terms of a cyclic decomposition of the primary module.

2. Ideals of Endomorphism Ring

In this section, we present a characterization of ideals of the endomorphism ring of a module M according to its cyclic decomposition. Based on the above description, we obtain a general form of ideals in the ring $End_R(M)$ as shown in the following theorem.

Theorem 1. *A nonempty subset $I \subseteq End_R(M)$ is an ideal of $End_R(M)$ if and only if*

$$I = \oplus_{i=1}^{m} \oplus_{j=1}^{m} \langle p^{k_{ij}} \delta_{ij} \rangle, \tag{5}$$

where $\delta_{ij} \in End_R(M)$ defined in equation (3), *for some nonnegative integers $k_{ij} \leq \min\{e_i, e_j\}$ satisfying*:

1. $k_{st} \leq k_{tt}$ *and* $k_{st} \leq k_{ss}$,

2. $k_{rt} \leq k_{st}$ *and* $k_{rt} \leq k_{rs}$ *for* $r \leq s \leq t$ *or* $r \geq s \geq t$,

3. $k_{ss} - k_{tt} \leq e_s - e_t$ *for* $s \leq t$,

where $r, s, t \in \{1, ..., m\}$.

Proof. Let $I \subseteq End_R(M)$ be an ideal. Consider a cyclic decomposition of M of the form (1). Thus we obtain the endomorphism ring $End_R(M)$ in the form (2), where $\delta_{ij} \in End_R(M)$ is defined as in (3). Using the fact that for every $i, j \in \{1, ..., m\}$,

$$\delta_{ii} I \delta_{jj} \subseteq I \cap \langle \delta_{ij} \rangle,$$

we obtain

$$I = \oplus_{i=1}^{m} \oplus_{j=1}^{m} \langle p^{k_{ij}} \delta_{ij} \rangle$$

for some nonnegative integers k_{ij}, $i, j \in \{1, ..., m\}$.

To show Property 1, let $s, t \in \{1, ..., m\}$ and $p^{k_{ss}} \delta_{ss}, p^{k_{tt}} \delta_{tt} \in I$. Then

$$\delta_{st} p^{k_{tt}} \delta_{tt} = p^{k_{tt}} \delta_{st} \in I$$

and

$$p^{k_{ss}} \delta_{ss} \delta_{st} = p^{k_{ss}} \delta_{st} \in I.$$

These imply that $p^{k_{st}} \mid p^{k_{tt}}$ and $p^{k_{st}} \mid p^{k_{ss}}$ for every $s, t \in \{1, ..., m\}$, and therefore

$$k_{st} \leq k_{tt} \text{ and } k_{st} \leq k_{ss}. \tag{6}$$

To show Property 2, let $\delta_{rs}, \delta_{st} \in End_R(M)$ and $p^{k_{st}} \delta_{st}, p^{k_{rs}} \delta_{rs} \in I$, where $r \leq s \leq t$ or $r \geq s \geq t$. Then according to (4), we have

$$\delta_{rs}(p^{k_{st}} \delta_{st}) = p^{k_{st}} \delta_{rt} \in I \text{ and } (p^{k_{rs}} \delta_{rs}) \delta_{st} = p^{k_{rs}} \delta_{rt} \in I.$$

These imply $p^{k_{rt}} \mid p^{k_{st}}$ and $p^{k_{rt}} \mid p^{k_{rs}}$, and therefore

$$k_{rt} \leq k_{st} \text{ and } k_{rt} \leq k_{rs}, \text{ where } r \leq s \leq t \text{ or } r \geq s \geq t. \tag{7}$$

To show Property 3, let $p^{k_{st}} \delta_{st}, p^{k_{ts}} \delta_{ts} \in I$, where $s \leq t$. Then

$$p^{k_{st}} \delta_{st} \delta_{ts} = p^{k_{st} + e_s - e_t} \delta_{ss}.$$

These imply $p^{k_{ss}} \mid p^{k_{st} + e_s - e_t}$, and hence $k_{ss} \leq k_{st} + e_s - e_t$ or $k_{ss} - e_s \leq$

$k_{st} - e_t$. Because of (6), we have $k_{ss} - e_s \le k_{st} - e_t \le k_{tt} - e_t$, and hence

$$k_{ss} - e_s \le k_{tt} - e_t, \text{ where } s \le t. \tag{8}$$

Conversely, if $I = \oplus_{i=1}^{m} \oplus_{j=1}^{m} \langle p^{k_{ij}} \delta_{ij} \rangle$, then I is a subgroup of $End_R(M)$. Since I satisfies (6), (7) and (8), for any $\delta_{st} \in End_R(M)$, where $s, t \in \{1, 2, 3, ..., m\}$ and $p^{k_{ij}} \delta_{ij} \in I$, we have

$$\delta_{st}(p^{k_{ij}} \delta_{ij}) \in I \text{ and } (p^{k_{ij}} \delta_{ij}) \delta_{st} \in I.$$

Hence I is an ideal of $End_R(M)$. □

Based on the above theorem we can identify some ideals in $End_R(M)$. For example:

(1) Consider a cyclic decomposition of M of the form (1), then for all $k \le e$, $p^k End_R(M)$ is an ideal in $End_R(M)$.

(2) Let $M = \langle u_1 \rangle \oplus \langle u_2 \rangle$ be a primary R-module. Then

$$\langle \delta_{11} \rangle = span\{\delta_{11}, \delta_{12}, \delta_{21}, p^{e_1 - e_2} \delta_{22}\} \subseteq End_R(M)$$

and

$$\langle \delta_{22} \rangle = span\{p^{e_1 - e_2} \delta_{11}, \delta_{12}, \delta_{21}, \delta_{22}\} \subseteq End_R(M)$$

are ideals in $End_R(M)$. Particularly $k_{ij} = 0$ in $\langle \delta_{11} \rangle$ except for k_{22} and $k_{ij} = 0$ in $\langle \delta_{22} \rangle$ except for k_{11}.

3. Fully Invariant Submodules

In this section we discuss the structure of fully invariant submodules in terms of the cyclic decomposition (1).

Definition 1. Let M be an R-module. A submodule X of M is called *fully invariant* if for every $f \in End_R(M)$, the condition $f(X) \subset X$ holds.

According to the definition, aM is a fully invariant submodule of M for any $a \in R$. We present a connection between fully invariant submodules and ideals of $End_R(M)$.

Lemma 1. *Let* $I \subseteq End_R(M)$ *be an ideal of the form* (5), X *be a submodule of* M, *and* $I(X) = \left\{\sum_{i=1}^{n} g_i x_i \mid g_i \in I,\, x_i \in X\right\}$. *Then* $I(X)$ *is a fully invariant submodule of* M.

Proof. Note that if $f \in End_R(M)$ and $v = \sum_{i=1}^{k} g_i x_i \in I(X)$ for some $g_i \in I,\, x_i \in X,$ then

$$f(v) = f\left(\sum_{i=1}^{k} g_i x_i\right) = \sum_{i=1}^{k} (fg_i) x_i \in I(X). \qquad \square$$

Conclude that if I is an ideal of $End_R(M)$, then IM is a fully invariant submodule of M. In general, a fully invariant submodule can be expressed as a direct sum of multiple of cyclic summands of the module as described in the following theorem.

Theorem 2. *Consider a cyclic decomposition of a module M as shown in* (1). *A submodule X of M is fully invariant if and only if*

$$X = \langle p^{s_1} u_1 \rangle \oplus \cdots \oplus \langle p^{s_m} u_m \rangle \tag{9}$$

for some $0 \le s_i \le e_i$, *where for each i with* $e_i \ge e_j$,

$$0 \le s_i - s_j \le e_i - e_j. \tag{10}$$

Proof. Let X be a fully invariant submodule of M and δ_{ij} for $i,\, j = 1, \ldots, m$ be the endomorphisms defined in (3). It is clear that $\delta_{ii}(X)$ is a submodule of $\langle u_i \rangle$, and hence

$$\delta_{ii}(X) = \langle p^{s_i} u_i \rangle$$

for some nonnegative integer s_i satisfying $0 \le s_i \le e_i$. Since X is fully invariant, we obtain $\langle p^{s_i} u_i \rangle = \delta_{ii}(X) \subseteq X$ for all $i = 1, ..., m.$ As a result

$$\langle p^{s_1} u_1 \rangle \oplus \cdots \oplus \langle p^{s_m} u_m \rangle \subseteq X.$$

Since $\sum_{i=1}^m \delta_{ii}$ is the identity map, we obtain

$$X = \left(\sum_{i=1}^m \delta_{ii} \right)(X) \subseteq \sum_{i=1}^m \delta_{ii}(X) = \langle p^{s_1} u_1 \rangle \oplus \cdots \oplus \langle p^{s_m} u_m \rangle.$$

Thus (9) holds. To show that (10) holds, let $e_i \geq e_j.$ Then

$$\delta_{ij} \langle p^{s_j} u_j \rangle = p^{e_i - e_j + s_j} u_i \in X \cap \langle u_i \rangle = \langle p^{s_i} u_i \rangle$$

and

$$\delta_{ji} \langle p^{s_i} u_i \rangle = p^{s_i} u_j \in X \cap \langle u_j \rangle = \langle p^{s_j} u_j \rangle.$$

These imply

$$0 \leq s_i \leq e_i - e_j + s_j \text{ and } 0 \leq s_j \leq s_i. \tag{11}$$

Therefore, (10) holds.

Conversely, let X be a submodule that satisfies (9) and (10). Let $i, j = 1, ..., m$ and suppose $e_i \geq e_j.$ Then

$$\delta_{ij}(p^{s_j} u_j) = p^{e_i - e_j + s_j} u_i \text{ and } \delta_{ji}(p^{s_i} u_i) = p^{s_i} u_j.$$

On the other hand, (10) implies (11). Hence

$$\delta_{ij}(p^{s_j} u_j) = p^{e_i - e_j + s_j} u_i \in \langle p^{s_i} u_i \rangle \subset X$$

and

$$\delta_{ji}(p^{s_i} u_i) = p^{s_i} u_j \in \langle p^{s_j} u_j \rangle \subset X.$$

Since for $i, j = 1, ..., m,$ δ_{ij} generate $End_R(M)$, we obtain that X is a fully invariant submodule. □

According to Theorem 2, we can describe the relationship between fully invariant submodules of a primary module with its cyclic decomposition.

Lemma 2. *Let* $I \subseteq End_R(M)$ *be an ideal of the form* (5), *and*

$$X = \oplus_{i=1}^{m} \langle p^{s_i} u_i \rangle$$

be a fully invariant submodule of M. Then

$$I(X) = \oplus_{i=1}^{m} \langle p^{\theta_i} u_i \rangle, \tag{12}$$

where $\theta_i = \min\{\theta_{il} \mid l \in \{1, 2, 3, ..., m\}\}$ *and*

$$\theta_{il} = \begin{cases} k_{il} + s_l, & \text{for } l \leq i, \\ k_{il} + s_l + e_i - e_l, & \text{for } l > i. \end{cases}$$

Proof. Let $p^{k_{ij}} \delta_{ij} \in I$ and $p^{s_l} u_l \in X$, with $i, j, l \in \{1, 2, 3, ..., m\}$. Then

$$(p^{k_{ij}} \delta_{ij})(p^{s_l} u_l) = \begin{cases} p^{k_{il} + s_l} u_i, & \text{for } l = j, l \leq i, \\ p^{k_{il} + s_l + e_i - e_l} u_i, & \text{for } l = j, l > i, \\ 0, & \text{for } l \neq j. \end{cases}$$

Let $\theta_i = \min\{\theta_{il} \mid l \in \{1, 2, 3, ..., m\}\}$, and

$$\theta_{il} = \begin{cases} k_{il} + s_l, & \text{for } l \leq i, \\ k_{il} + s_l + e_i - e_l, & \text{for } l > i. \end{cases} \tag{13}$$

Then

$$(p^{k_{ij}} \delta_{ij})(p^{s_l} u_l) \in \langle p^{\theta_i} u_i \rangle.$$

Furthermore, because $I = \oplus_{i=1}^{m} \oplus_{j=1}^{m} \langle p^{k_{ij}} \delta_{ij} \rangle$ and $X = \oplus_{i=1}^{m} \langle p^{s_i} u_i \rangle$, we have

$$I(X) = \left\{ \sum_{i=1}^{n} g_i x_i \mid g_i \in I, x_i \in X \right\} \subseteq \oplus_{i=1}^{m} \langle p^{\theta_i} u_i \rangle.$$

Conversely, if $x \in \oplus_{i=1}^{m} \langle p^{\theta_i} u_i \rangle$, then $x = \sum_{i=1}^{m} \alpha_i p^{\theta_i} u_i$ for $\alpha_i \in R$. Because of (13), there exists an l_0 such that $\theta_i = \theta_{il_0}$. Furthermore, if $i \geq l_0$, then

$$x = \sum_{i=1}^{m} \alpha_i p^{\theta_{il_0}} u_i = \sum_{i=1}^{m} \alpha_i p^{k_{il_0} + t_{l_0}} u_i = \sum_{i=1}^{m} \alpha_i (p^{k_{il_0}} \delta_{il_0})(p^{t_{l_0}} u_{l_0}) \in I(X)$$

and if $i < l_0$, then

$$x = \sum_{i=1}^{m} \alpha_i p^{\theta_{il_0}} u_i = \sum_{i=1}^{m} \alpha_i p^{k_{il_0} + t_{l_0} + e_i - el_0} u_i$$

$$= \sum_{i=1}^{m} \alpha_i (p^{k_{il_0}} \delta_{il_0})(p^{t_{l_0}} u_{l_0}) \in I(X). \qquad \square$$

According to Lemma 2, we can describe the form of $I(X)$ by its cyclic decomposition. Note that for every fully invariant submodule $X \subseteq M$, there exists an $f \in End_R(M)$ such that $f(M) = X$. Furthermore, we can see that for every fully invariant submodule $X \subseteq M$, there exists an ideal $I \subseteq End_R(M)$ such that $I(M) = X$.

According to Theorem 2, we can see the connection between a fully invariant submodule in M with its direct summands.

Corollary 1. *Consider a cyclic decomposition of a module M as shown in* (1). *Let H be a submodule fully invariant of* $\langle u_i \rangle$. *Then there exists a fully invariant submodule* $\overline{H}$ *of M with H as a direct summand of* $\overline{H}$.

Proof. Let $H = \langle p^{h_i} u_i \rangle$, $0 \leq h_i \leq e_i$ be a fully invariant submodule of $\langle u_i \rangle$. For $j = 1, \ldots, m$, let

$$h_j = \begin{cases} h_i, & \text{if } j \leq i, \\ \min\{h_i, e_j\}, & \text{if } j > i. \end{cases}$$

Then $h_1, h_2, \ldots, h_m$ satisfy conditions

$$0 \leq h_i \leq e_i, \text{ for all } i = 1, \ldots, m$$

and

$$0 \leq h_i - h_j \leq e_i - e_j, \text{ if } e_i \geq e_j.$$

Thus the submodule

$$\overline{H} = \bigoplus_{i=1}^{n} \langle p^{h_i} u_i \rangle$$

according to Theorem 2 is fully invariant in M with H as one of its direct summands. □

4. S-prime Submodules

In this section we describe a characterization of S-prime submodules in primary modules. The concept of an S-prime submodule was introduced by Sanh et al. as a generalization of prime ideals in an associative ring [2].

Definition 2. A proper fully invariant submodule $X \subseteq M$ is called an *S-prime submodule* of M if for every ideal I of $End_R(M)$ and every fully invariant submodule U of M satisfying $I(U) \subset X$, we have $U \subseteq X$ or $I(M) \subseteq X$.

Our characterization of an S-prime submodule is developed in relation with the characterization of fully invariant submodules in terms of the cyclic decomposition of the module.

Theorem 3. *Consider a cyclic decomposition of a module M as shown in* (1), *and let* i_0 *be the smallest positive integer* $i_0 \in \{1, 2, 3, \ldots, m\}$ *such that* $e_1 > e_{i_0}$. *Let X be a proper fully invariant submodule of M which satisfies* (9) *and* (10). *Then X is an S-prime submodule of M if and only if* $0 \leq s_i \leq 1$, *where* $s_1 = 1$ *and* $s_{i_0} = 0$.

Proof. Let X be an S-prime submodule of M. Suppose there exists $1 \leq i \leq m$ such that $s_i \geq 2$. Since the integers $s_1, ..., s_m$ satisfy (10), we have $s_1 \geq 2$. Let $i_1 > 1$ be the smallest positive integer such that $s_1 > s_{i_1}$. That means the following inequalities

$$s_1 = \cdots = s_{i_1 - 1} > s_{i_1} \geq s_{i_1 + 1} \geq \cdots \geq s_m \geq 0$$

hold. Define m-tuple of nonnegative integers $(t_1, ..., t_m)$ as follows:

$$t_i = \begin{cases} s_i - 1, & \text{if } i = 1, ..., i_1 - 1, \\ s_i, & \text{if } i = i_1, i_1 + 1, ..., m. \end{cases}$$

It is easy to verify that $(t_1, ..., t_m)$ satisfies

$$0 \leq t_i - t_j \leq e_i - e_j \quad \text{for each } e_i \geq e_j$$

and so the submodule $U = \bigoplus_{i=1}^{m} \langle p^{t_i} u_i \rangle$ is fully invariant but $U \not\subseteq X$ since $s_1 > t_1$. Let $I = pEnd_R(M)$ be the ideal of $End_R(M)$ generated by the endomorphism $p\delta_{ij}$, where $\delta_{ij} \in End_R(M)$ is defined in (3). We obtain $I(U) \subseteq X$, but $U \not\subseteq X$ and $I(M) = pM \not\subseteq X$ since $pu_1 \in pM$ and $pu_1 \notin X$. This contradicts the fact that X is S-prime. Thus $0 \leq s_i \leq 1$ for all $i = 1, ..., m$.

Furthermore, let $i_0 \in \{1, 2, 3, ..., m\}$ be the smallest positive integer such that $e_1 > e_{i_0}$. Suppose that $s_1 = s_{i_0} = 1$ and $i_1 > i_0$. That means the following inequalities hold:

$$s_1 = \cdots = s_{i_0 - 1} = s_{i_0} = s_{i_0 + 1} = \cdots = s_{i_1 - 1} > s_{i_1} \geq s_{i_1 + 1} \geq \cdots \geq s_m \geq 0.$$

Define the ideal $I = \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{m} \langle p^{k_{ij}} \delta_{ij} \rangle \subseteq End_R(M)$, where

$$k_{ij} = \begin{cases} 0, & \text{for } 1 \leq i < i_0 \text{ or } 1 \leq j < i_0, \\ 1, & \text{for } i \geq i_0 \text{ and } j \geq i_0, \end{cases} \tag{14}$$

and a fully invariant submodule $U = \oplus_{i=1}^{m}\langle p^{t_i}u_i\rangle \subseteq M$, where

$$t_i = \begin{cases} s_i, & \text{for } 1 \leq i < i_0, \\ 0, & \text{for } i_0 \leq i \leq m. \end{cases} \tag{15}$$

Since $0 = t_{i_0} < s_{i_0} = 1$, $U \nsubseteq X$. Note that

$$I(U) = \oplus_{i=1}^{m}\langle p^{\theta_i}u_i\rangle,$$

where $\theta_i = \min\{\theta_{il} \mid l \in \{1, 2, 3, ..., m\}\}$ with

$$\theta_{il} = \begin{cases} k_{il} + t_l, & \text{for } l \leq i, \\ k_{il} + t_l + e_i - e_l, & \text{for } l > i. \end{cases}$$

From (14) and (15), for $1 \leq i < i_0$,

$$\theta_{il} = \begin{cases} k_{il} + t_l = 1, & \text{for } l \leq i, \\ k_{il} + t_l + e_i - e_l = 1, & \text{for } l > i \text{ and } l < i_0, \\ k_{il} + t_l + e_i - e_l = e_i - e_l, & \text{for } l > i \text{ and } l \geq i_0, \end{cases}$$

and for $i_0 \leq i \leq m$,

$$\theta_{il} = \begin{cases} k_{il} + t_l = 1, & \text{for } l \leq i \text{ and } l \geq i_0, \\ k_{il} + t_l = 2, & \text{for } l \leq i \text{ and } l < i_0, \\ k_{il} + t_l + e_i - e_l = 1 + e_i - e_l, & \text{for } l > i \text{ and } l > i_0, \end{cases}$$

which imply $\theta_i = 1$. Hence $I(U) \subseteq X$.

On the other hand

$$I(M) = \oplus_{i=1}^{m}\langle p^{\eta_i}u_i\rangle,$$

where $\eta_i = \min\{\eta_{il} \mid l \in \{1, 2, 3, ..., m\}\}$ with

$$\eta_{il} = \begin{cases} k_{il}, & \text{for } l \leq i, \\ k_{il} + e_i - e_l, & \text{for } i < l. \end{cases}$$

From (14), for $1 \leq i < i_0$,

$$\eta_{il} = \begin{cases} k_{il} = 0, & \text{for } l \leq i, \\ k_{il} + e_i - e_l = 0, & \text{for } l > i \text{ and } l < i_0, \\ k_{il} + e_i - e_l = e_i - e_l, & \text{for } l > i \text{ and } l \geq i_0, \end{cases}$$

and for $i_0 \leq i \leq m$,

$$\eta_{il} = \begin{cases} k_{il} = 1, & \text{for } l \leq i \text{ and } l \geq i_0, \\ k_{il} = 0, & \text{for } l \leq i \text{ and } l < i_0, \\ k_{il} + e_i - e_l = 1 + e_i - e_l, & \text{for } l > i \text{ and } l > i_0, \end{cases}$$

which imply $\eta_i = 0$. Hence $I(M) = M \nsubseteq X$. Thus $I(U) \subseteq X$ but $U \nsubseteq X$ and $I(M) = M \nsubseteq X$. This contradicts the fact that S is an S-prime. Thus $s_{i_0} \neq s_1$ or $s_{i_0} = 0$.

Conversely, let X be of the form (9) and (10) with $s_1 = 1$ and $s_{i_0} = 0$. Let I be an ideal of $End_R(M)$ of the form (5) and $U = \oplus_{i=1}^{m} \langle p^{t_i} u_i \rangle$ be a fully invariant submodule of M such that $I(U) \subseteq X$. Consider that

$$I(U) = \oplus_{i=1}^{m} \langle p^{\theta_i} u_i \rangle, \tag{16}$$

where $\theta_i = \min\{\theta_{il} \mid l \in \{1, 2, 3, ..., m\}\}$ with

$$\theta_{il} = \begin{cases} k_{il} + t_l, & \text{for } l \leq i, \\ k_{il} + t_l + e_i - e_l, & \text{for } i < l, \end{cases}$$

and

$$I(M) = \oplus_{i=1}^{m} \langle p^{\eta_i} u_i \rangle, \tag{17}$$

where $\eta_i = \min\{\eta_{il} \mid l \in \{1, 2, 3, ..., m\}\}$ with

$$\eta_{il} = \begin{cases} k_{il}, & \text{for } l \leq i, \\ k_{il} + e_i - e_l, & \text{for } i < l. \end{cases}$$

Suppose $U \nsubseteq X$, since $s_1 = 1$ and $s_1 > s_{i_0}$ for $i_1, i_0 \in \{1, ..., m\}$, we obtain $t_1 = 0$ and $s_1 = 1$. According to (10), we have that $t_i = 0$ for $i \in \{1, 2, ..., m\}$, furthermore

$$\theta_{il} = \begin{cases} k_{il}, & \text{for } l \leq i, \\ k_{il} + e_i - e_l, & \text{for } i < l. \end{cases}$$

Hence $\theta_i = \eta_i$ and

$$I(M) = I(U) \subseteq X.$$

Thus X is S-prime. □

According to above theorem we can easily identify an S-prime submodule in a primary module. For example: Consider $M = Z_{64} \oplus Z_{16} \oplus Z_8$ as a Z-module, then $2Z_{64} \oplus 2Z_{16} \oplus Z_8$ is not an S-prime submodule of M.

We close this section with a description of a relation between prime submodules and S-prime submodules.

Theorem 4. *Let M be a primary R-module. If $X \subseteq M$ is an S-prime submodule of M, then X is a prime submodule of M.*

Proof. Consider a cyclic decomposition of the module M as shown in (1) and so the submodule X is of the form (9) which satisfies (10) with $0 \leq s_i \leq 1$, where $s_1 = 1$ and $s_1 > s_{i_0}$.

We prove that X is a prime submodule of M. From the fact $0 \leq s_i \leq 1$ and $s_1 = 1$, we obtain that the ideal

$$(X : M) = \{r \in R : rM \subseteq X\}$$

is generated by p. Let $r \in R$, $x \in M$ be such that $rx \in X$. Let $x = \sum_{i=1}^{m} \alpha_i u_i$ for some $\alpha_i \in R$. Then $rx = \sum_{i=1}^{m} r\alpha_i u_i$. Suppose $x \notin X$. Then X is prime if we can show that $r \in (X : M)$ which is obtained by showing that $p \mid r$. From $x \notin X$, X is of the form (9), and $x = \sum_{i=1}^{m} \alpha_i u_i$; there exists $i_1 < i_0$ such that $\alpha_{i_1} u_{i_1} \notin \langle p^{s_{i_1}} u_{i_1} \rangle$. This implies $s_{i_1} = 1$ and $p \nmid \alpha_{i_1}$. Because $rx \in X$, we obtain $r\alpha_i u_i \in \langle p^{s_i} u_i \rangle$ for all $i = 1, \ldots, m$. In particular, we have $r\alpha_{i_1} u_{i_1} \in \langle p u_{i_1} \rangle$. Thus $p \mid r\alpha_{i_1}$. Because p is prime and $p \nmid \alpha_{i_1}$, then $p \mid r$. □

Note that the converse of the above theorem is not true. There exists a prime submodule which is not fully invariant. Hence not every prime submodule is an S-prime submodule. For example, consider $M = \mathbb{Z}_8 \times \mathbb{Z}_{32}$ as a $\mathbb{Z}$-module. Then $X = 0 \times 2\mathbb{Z}_{32}$ is a prime submodule. This is because for every $m = (a, b) \in M$, $r \in \mathbb{Z}$ with $rm = (ra, rb) \in X$ implies that $r \in (X : M)$ or $m \in X$. In contrast the submodule X is not S-prime, because X is not fully invariant.

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