JP Journal of Algebra, Number Theory and Applications

Published Online: September 2016
http://dx.doi.org/10.17654/NT038050499
Volume 38, Number 5, 2016, Pages 499-506 **ISSN: 0972-5555**

FINITISTIC *n*-SELF-COTILTING MODULES OVER RING EXTENSIONS

Salah Al-Nofayee

Department of Mathematics
Taif University
Taif, Saudi Arabia
e-mail: alnofayee@hotmail.com

Abstract

For a ring A, a ring extension B, and a fixed right A-module M, we give conditions under which a finitistic n-self-cotilting module M extends to a finitistic n-self-cotilting module $K = Hom_A(B, M)$ over the ring extension B. We prove that in case there is a ring homomorphism $\beta : B \to A$ (hence A and B are ring extensions of each other), K is M-reflexive and $M \sim K$ (that is $K|M$ in Mod-A and $M|K$ in Mod-B), then M is finitistic n-self-cotilting module if and only if the induced module $K = Hom_A(B, M)$ is finitistic n-self-cotilting module.

1. Introduction

Over the years, many authors introduced various generalizations of Morita duality. For example, quasi-duality, cotilting duality, generalized and weak Morita duality and dualities induced by costar and (f)-cotilting

Received: April 25, 2016; Revised: May 17, 2016; Accepted: July 10, 2016
2010 Mathematics Subject Classification: 16D90.
Keywords and phrases: finitistic n-self-cotilting modules, ring extensions.
Communicated by K. K. Azad

modules. As a continuation of this work Breaz in [3], studied a duality induced by finitistic n-self-cotilting module.

In [6], Fuller showed that some properties related to Morita duality pass from a right A-module M to the induced B-module $Hom_A(B, M)$, where B is a ring extension of the ring A. For example injective cogenerator, Morita duality, quasi-duality module, cotilting module. Recently the author, in [1], proves that the property of being costar module or r-costar module is inherited by $Hom_A(B, M)$ from M. On the other side it has been shown by many researchers that some properties related to Morita equivalence pass from a right A-module M to the induced B-module $M \otimes_A B$, where B is a ring extension of the ring A. For example projective, generator, finitely generated, tilting module quasi-progenerator and $*$-module. See for examples [2, 5, 7].

Following a preliminary section, we prove in Section 2 our main theorem, Theorem 10. In that theorem, under certain conditions, we show that M is finitistic n-self-cotilting module if and only if $Hom_A(B, M)$ is finitistic n-self-cotilting module. In order to show this main result we have to prove some minor results. For example we prove in Corollary 8, that $n\text{-}cop(M) = (n+1)\text{-}cop(M)$ if and only if $n\text{-}cop(K) = (n+1)\text{-}cop(K)$.

We assume that the rings are associative with identity, the ring homomorphisms are identity preserving, all (left, right) modules are unital, and all subcategories are full and additive.

2. Preliminaries

Let U be an arbitrary A-module, where A is a ring. A right module Q is said to be *n-finitely U-copresented*, where n is a positive integer, if there exists a long exact sequence

$$0 \to Q \to U^{m_0} \to U^{m_1} \to \cdots \to U^{m_{n-1}},$$

such that m_i are positive integers, for all $0 \le i \le n-1$. The class of all n-finitely U-copresented modules is denoted by $n\text{-}cop(U)$. Note that the class 1-$cop(U)$ is the class of all finitely U-cogenerated modules and it is denoted by $cog(U)$. It is clear that $(n+1)$-$cop(U) \subseteq n$-$cop(U)$ for every n. A left A-module Q is called *n-finitely* presented if there exists an exact sequence

$$A^{m_{n-1}} \to \cdots \to A^{m_1} \to A^{m_0} \to Q \to 0,$$

such that m_i are positive integers, for all $0 \leq i \leq n-1$. The class of all n-finitely presented modules is denoted by $FP_{(n)}(A)$. Note that the class $FP_{(1)}(A)$ is the class of all finitely generated left A-modules.

Let A be any ring, M be a fixed right A-module and $D = End_A(M)$. Let B be another ring and $\alpha : A \to B$ a ring homomorphism, then we say that B is a ring extension of A. It is clear that every right B-module can be considered as a right A-module. Moreover $Hom_B(N, L) \subseteq Hom_A(N, L)$, for every right B-modules N and L. Suppose that $K = Hom_A(B, M)$ and $E = End_B(K)$, then the ring homomorphism $\gamma : D \to E$ defined via $\gamma(d)(f) = d \circ f$, for all $d \in D$ and $f \in K$, is clearly identity preserving. If there is another homomorphism $\beta : B \to A$, then A and B are ring extension of each other, and hence D and E are ring extension of each other.

We will denote the functors $Hom_A(-, M)$ and $Hom_D(-, M)$ by Δ_M and the functors $Hom_B(-, K)$ and $Hom_E(-, K)$ by Δ_K. As in [3], a right module M is called *finitistic* n-self-cotilting module if for any exact sequence

$$0 \to L \to M^m \to N \to 0,$$

such that $N \in n$-$cop(M)$ and m is a positive integer, then

$$0 \to \Delta_M(N) \to \Delta_M(M^m) \to \Delta_M(L) \to 0$$

is exact, and n-$cop(M) = (n+1)$-$cop(M)$.

We say that N weakly divide M in $M^m \cong N \oplus Q$, for some positive integer m and for some $Q \in Mod$-A and we write $N \mid M$.

For a right A-module N we define

$$^{\perp<n}N = \{Q \in Mod\text{-}A : Ext_A^i(Q, N) = 0 \text{ for all } 0 < i < n\}.$$

Consider a bimodule ${}_DM_A$. For a right A-module N_A, if the evaluation map $\delta_N : N_A \to Hom_D(Hom_A(N, M), M)$ is an isomorphism then we say that N is M-reflexive.

From now on let A and B be two rings such that B is a ring extension of A, M be a fixed right A-module, $D = End_A(M)$, $K = Hom_A(B, M)$ and $E = End_B(K)$.

Proposition 1 [6, Proposition 3.3]**.** *Suppose that* V_A *is a right B-module and W is a left E-module. If K is M-reflexive, then*

1. V_B *is K-reflexive if and only if* V_A *is M-reflexive*;

2. ${}_EW$ *is K-reflexive if and only if* ${}_DW$ *is M-reflexive*.

Lemma 2 [6, Lemma 1.3 and Lemma 3.2]**.** (i) *For any* $V \in Mod\text{-}B$,

$$Hom_B(V, K) \cong Hom_A(V, M).$$

As a special case, for K, we have $E = Hom_B(K, K) \cong Hom_A(K, M)$.

(ii) *if K is M-reflexive, then for any* $W \in E\text{-}Mod$, *we have*

$$Hom_E(W, K) \cong Hom_D(W, M).$$

Remark 3. If A and B are ring extensions of each other, then any projective B-module is projective as an A-module and vise versa. If P is a projective resolution of the B-module N, then P is a projective resolution of the A-module N and vise versa. We can say the same thing about projective modules and projective resolutions in D-*Mod* and E-*Mod*.

Lemma 4 [4, Lemma 4.2.4]**.** *Let* $0 \to N \xrightarrow{f} Q \to L \to 0$ *be a short exact sequence in Mod-A. If Q is M-reflexive and* $L \in cog(M)$, *then* $\Delta_M(f)$ *is an epimorphism if and only if* $\operatorname{Im}\Delta_M(f)$ *is M-reflexive*.

Theorem 5 [3, Theorem 2.7]**.** *M is finitistic n-self-cotilting module if and only if there is a duality*

$$\Delta_M : n\text{-}cop(M) \leftrightarrows {}^{\perp<n}M \cap FP_{(n+1)}(D) \cap cog({}_D M).$$

Lemma 6. *If* $K \mid M$ *in Mod-A, then E is finitely generated in D-Mod.*

Proof. Since $K \mid M$ in *Mod-A*, $M^n \cong K \oplus Q$, for some positive integer n and for some $Q \in Mod\text{-}A$. Applying the functor $Hom_A(-, M)$ we conclude that *E* is finitely generated in *D-Mod*. □

3. Finitistic *n*-self-cotilting Module

We assume throughout this section that *K* is *M*-reflexive. Also we suppose that there is a ring homomorphism $\beta : B \to A$ and hence *A* and *B* are ring extensions of each other.

Lemma 7. *For any right B-module N,* $N \in n\text{-}cop(M)$ *if and only if* $N \in n\text{-}cop(K)$.

Proof. Let $N \in n\text{-}cop(M)$. We have an exact sequence

$$0 \to N \to M^{m_0} \to M^{m_1} \to \cdots \to M^{m_{n-1}},$$

where m_i are positive integers for all $0 \le i \le n-1$. Applying the functor $Hom_A(B, -)$, we get an exact sequence

$$0 \to Hom_A(B, N) \to K^{m_0} \to K^{m_1} \to \cdots \to K^{m_{n-1}}.$$

Since $N \cong Hom_B(B, N) = Hom_A(B, N)$, $N \in n\text{-}cop(K)$. Conversely, let $N \in n\text{-}cop(K)$. We have an exact sequence

$$0 \to N \to K^{m_0} \to K^{m_1} \to \cdots \to K^{m_{n-1}},$$

applying the functor $Hom_B(A, -)$, we get an exact sequence

$$0 \to N \to M^{m_0} \to M^{m_1} \to \cdots \to M^{m_{n-1}},$$

since $N \cong Hom_A(A, N) \cong Hom_B(A, N)$, and $Hom_B(A, K) = Hom_B(A, Hom_A(B, M)) \cong Hom_A(A \otimes_B B, M) \cong M$. Hence $N \in n\text{-}cop(M)$. $\square$

Corollary 8. *$n\text{-}cop(M) = (n+1)\text{-}cop(M)$ if and only if $n\text{-}cop(K) = (n+1)\text{-}cop(K)$.*

In the following proposition, we use the same techniques of [3, Lemma 2.6].

Proposition 9. *Suppose that $K \mid M$ in Mod-A. Let M be a finitistic n-self-cotilting module. If we have a short exact sequence*

$$0 \to L \xrightarrow{f} K^n \xrightarrow{g} N \to 0,$$

where n is a positive integer, and $N \in n\text{-}cop(M)$, then $\Delta_M(f)$ is an epimorphism.

Proof. Applying the functor Δ_M to the sequence $0 \to L \xrightarrow{f} K^n \xrightarrow{g} N \to 0$ we get following short exact sequence

$$0 \to \Delta_M(N) \to \Delta_M(K^n) \to Q \to 0, \tag{1}$$

where $Q = \operatorname{Im} \Delta_M(f)$. By the duality in Theorem 5, it is clear that $\Delta_M(N) \in FP_{(n+1)}(D)$. Since $\Delta_M(K^n) = Hom_A(K^n, M) \cong Hom_B(K^n, K) = E^n$ and E^n is finitely generated in D-Mod, by Lemma 6, we can get an exact sequence $0 \to \Delta_M(N) \to D^m \to Q \to 0$, where m is a positive integer. Hence it is clear that $Q \in FP_{(n+1)}(D)$. Since Q is a submodule of $\Delta_M(L) \in cog({}_D M)$, $Q \in cog({}_D M)$. Applying the functor Δ_M to the sequence (1), we get the following long exact sequence

$$\begin{aligned} 0 &\to \Delta_M(Q) \to \Delta_M^2(K^n) \xrightarrow{\Delta_M^2(g)} \Delta_M^2(N) \to Ext_D^1(Q, M) \\ &\to Ext_D^1(\Delta_M(K^n), M) \to Ext_D^1(\Delta_M(N), M) \to Ext_D^2(Q, M) \end{aligned}$$

$$\to Ext_D^2(\Delta_M(K^n), M) \to Ext_D^2(\Delta_M(N), M) \to \cdots. \tag{2}$$

By Lemma 2 and Remark 3, $Ext_D^i(\Delta_M(K^n), M) \cong Ext_E^i(\Delta_M(K^n), K) = Ext_E^i(E^n, K) = 0, \forall i \geq 1$. By assumptions and Theorem 5, $Ext_D^i(\Delta_M(N), M) = 0, \forall i \geq 1$. Thus by the exactness, $Ext_D^i(Q, M) = 0, \forall i \geq 2$. Now consider the following commutative diagram:

$$\begin{array}{ccccc} K^n & \xrightarrow{g} & N & \longrightarrow & 0 \\ \downarrow\cong & & \downarrow\cong & & \\ \Delta_M^2(K^n) & \xrightarrow[\Delta_M^2(g)]{} & \Delta_M^2(N) & & \end{array}.$$

Since g is an epimorphism and the vertical arrows are isomorphisms, it is clear that $\Delta_M^2(g)$ is an epimorphism, so $Ext_D^1(Q, M) = 0$. Hence $Q \in {}^{\perp<n}M$. Now since $Q \in {}^{\perp<n}M \cap FP_{(n+1)}(D) \cap cog({}_DM)$, by the duality in Theorem 5, Q is reflexive. Now by Lemma 4, $\Delta_M(f)$ is an epimorphism. $\square$

In the following theorem we suppose that $K \mid M$ in *Mod-A* and $M \mid K$ in *Mod-B* and we write $M \sim K$.

Theorem 10. *Let* $M \sim K$. *Then M is finitistic n-self-cotilting if and only if K is finitistic n-self-cotilting.*

Proof. Let M be a finitistic n-self-cotilting module. Suppose we have an exact sequence $0 \to L \xrightarrow{f} K^m \xrightarrow{g} N \to 0$, such that $N \in n\text{-}cop(K)$ and m is a positive integer. By Lemma 7, $N \in n\text{-}cop(M)$. Hence by Proposition 9, $\Delta_M(f)$ is epimorphism, therefore $0 \to \Delta_M(N) \xrightarrow{\Delta_M(g)} \Delta_M(K^m) \xrightarrow{\Delta_M(f)} \Delta_M(L) \to 0$ is exact in *D-Mod* and hence in *E-Mod*. By Corollary 8, $n\text{-}cop(K) = (n+1)\text{-}cop(K)$. Thus K finitistic n-self-cotilting module. If we suppose that K finitistic is n-self-cotilting module, then we can prove similarly that M is finitistic n-self-cotilting module keeping in mind that

$$\begin{aligned}
& Hom_B(A, K) \\
\cong\ & Hom_B(A, Hom_A(B, M)) \\
\cong\ & Hom_A(A \otimes_B B, M) \\
\cong\ & Hom_A(A, M) \\
\cong\ & M.
\end{aligned}$$

and M is K-reflexive by Proposition 1. □

Acknowledgement

The author thanks the anonymous referees for their valuable suggestions which led to the improvement of the manuscript.

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