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INFINITE GROUPS WITH TWO CONJUGACY CLASSES OF NON-SUBNORMAL SUBGROUPS

Aifang Feng and Zuhua Liu

Department of Mathematics
Kunming University
Kunming, 650214, P. R. China
e-mail: fengaifang888@sina.com
liuzuhua@hotmail.com

Abstract

Let G be an infinite group. μ denotes the number of the conjugacy classes of non-subnormal subgroups of G. $\mu_{<\infty}$ and μ_{∞} denote the number of the finitely length and infinitely length conjugacy classes of non-subnormal subgroups of G, respectively.

Let G be a group with $\mu = 2$. Then it is proved that

(1) there exists no infinite group with $\mu_{<\infty} = 2$;

(2) if $\mu_{<\infty} = \mu_{\infty} = 1$, then there exists some normal subgroup $N \lhd G$, such that G/N is a finite non-nilpotent inner-abelian group, where N is a group with all subgroups subnormal;

(3) if G is infinite locally finite, then G is a Baer group, and $\mu_{\infty} = 2$.

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1. Introduction

This paper continues to study the groups in which the set of non-subnormal subgroups is restricted in some way. It is well known that groups with all subgroups subnormal are locally nilpotent. More recently, we have the work of Möhres that every group with all subgroups subnormal is soluble. Now, there are various results on groups in which the set of non-subnormal subgroups is non-empty, but there is some restrictive condition on non-subnormal subgroups - for example [2, 5, 6, 7, 9]. On the other hand, many researchers have focused on the groups with finite conjugacy classes of some special subgroups, see [10, 11]. In [3], the authors have proved that groups with one finitely length conjugacy class of non-subnormal subgroups are finite non-nilpotent inner-abelian groups. Here we consider groups with two conjugacy classes of non-subnormal subgroups.

Let G be an infinite group. μ denotes the number of the conjugacy classes of non-subnormal subgroups of G. $\mu_{<\infty}$ and μ_{∞} denote the number of the finitely length and infinitely length conjugacy classes of non-subnormal subgroups of G, respectively.

2. Basic Definitions and Preliminaries

In this section, we first state some results which will be used in the sequel.

Lemma 1 [1]. *Let G be a finite group. Then* $\mu = 1$ *if and only if G is a finite non-nilpotent inner-abelian group, that is,*

$$\begin{aligned} G = \langle a, b_1, b_2, \ldots, b_\beta \mid a^{p^\alpha} = 1 = b_1^q = b_2^q = \cdots = b_\beta^q; \\ [b_i, b_j] = 1,\ i,\ j = 1, 2, \ldots, \beta;\ b_i^a = b_{i+1},\ i = 1, 2, \ldots, \beta - 1; \\ b_\beta^a = b_1^{d_1} b_2^{d_2} \cdots b_\beta^{d_\beta} \rangle, \end{aligned}$$

where $f(x) = x^\beta - d_\beta x^{\beta-1} - \cdots - d_2 x - d_1$ *is an irreducible polynomial* *over the field* $\mathbf{F}_q$, *which divides* $x^p - 1$, *and* $q^\beta \equiv 1 \pmod{p}$, $p \neq q$ *are prime numbers*.

Lemma 2 [8]**.** *Let* G *be a group*, $H, K \leq G$. *If* $H \operatorname{sn} G$, $K \operatorname{sn} G$, *and* K *normalizes* H, *then* $J = \langle H, K \rangle$ *is subnormal in* G.

Lemma 3 [8]**.** *Let* G *be a group*, $H \leq G$. *If* $|G : H| = n$, *then* $|G : H_G| \mid n!$.

Lemma 4 [8]**.** *Let* G *be a nilpotent group with nilpotent class* c. *If* $H < G$, *and* $H = H_0$, $H_{i+1} = N_G(H_i)$, *then* $H_c = G$.

Definition 1. Let $\mathscr{P}$ be a property of groups. Then a group G is called a *locally* $\mathscr{P}$*-group* if each finite subset of G is contained in a $\mathscr{P}$-subgroup of G. If the property $\mathscr{P}$ is inherited by subgroups, this is equivalent to the requirement that each finitely generated subgroup have $\mathscr{P}$.

Definition 2. Let G be a group and $B \leq G$. Then B is called the *Baer-radical* of G, if B is generated by all the abelian subnormal subgroups of G. And G is called a *Baer group*, if $G = B$, it is equivalent to the requirement that every cyclic subgroup of G is subnormal. Clearly, the Baer-radical B is the maximal Baer group of G. The Baer group is a special type of locally nilpotent groups.

Lemma 5 [4]**.** *Let* G *be a locally nilpotent group in which every subgroup that is not self-normalizing is subnormal. Then every subgroup of* G *is subnormal*.

Lemma 6. *There exists no locally nilpotent group with* $\mu_{<\infty} \geq 1$.

Proof. Let H be non-subnormal in G, and $|G : N_G(H)| = n < \infty$. Assume that G is locally nilpotent. Let $N = N_G(H)_G = \bigcap_{g \in G} (N_G(H))^g$. Then $N \lhd G$, and $|G : N| \mid n!$ by Lemma 3, hence G/N is finite. Since G is locally nilpotent, G/N is locally nilpotent. Thus G/N is nilpotent. It follows that $N_G(H)/N \operatorname{sn} G/N$ and $H \operatorname{sn} G$, a contradiction.

Remark. Lemma 6 shows that if H is a non-subnormal subgroup of the locally nilpotent group G, then H has infinitely conjugates in G.

Lemma 7 [4]. *Let G be a locally finite group and suppose that G is not locally nilpotent. Then every non-subnormal subgroup of G is self-normalizing if and only if the following hold.*

(1) $G = P \ltimes A$, *where* $P = \langle g \rangle$ *is a cyclic p-subgroup for some prime p.*

(2) *A is a nilpotent* p'*-subgroup and* $C_P(A) = \langle g^p \rangle$.

(3) $G' = A$.

(4) $C_G(P) = P$.

Lemma 8 [8]. *Let G be a locally finite and P be a finite Sylow p-subgroup of G. Then all Sylow p-subgroups of G are conjugate.*

3. Main Results

Theorem 1. *There exists no infinite group with* $\mu = \mu_{<\infty} = 2$.

Proof. Assume that G is an infinite group with $\mu = \mu_{<\infty} = 2$. Let H and K be non-subnormal and not conjugate in G. By Lemma 6, G is not locally nilpotent, and hence G is not a Baer group, which follows that H or K is a cyclic group. Let $H = \langle a \rangle$ be a cyclic group. Then H is a finite cyclic group. In fact, if H is infinite, then $\langle a^p \rangle$ is the maximal normal subgroup of H, for every prime p. Since G has only finitely non-subnormal subgroups, there exist at least two different maximal normal subgroups $\langle a^p \rangle$ and $\langle a^q \rangle$ $(p \neq q)$ of H, such that $\langle a^p \rangle \mathrm{sn} G$ and $\langle a^q \rangle \mathrm{sn} G$. Thus $H = \langle a^p, a^q \rangle \mathrm{sn} G$ by Lemma 2, a contradiction. So H is a finite cyclic group. Let $N = N_G(H)$. Then N is also non-subnormal in G, and N is conjugate to H or K. Since $|G : N| < \infty$, N is infinite, and hence N is conjugate to K. Without loss of generality, let $N = K$ and $H \lhd K$. Now, $N_G(K) = K$. In fact, since $N_G(K)$ is non-subnormal in G, $N_G(K)$ is conjugate to K. If $K < N_G(K)$, then $K = K_0 < K_1 < K_2 < \cdots < K_n < \cdots$ is a chain of non-subnormal subgroups of G, where $K_{i+1} = N_G(K_i)$, furthermore, K_i is conjugate to K for all i, which contradicts $|G : N_G(K)| < \infty$. Let $K_G = \bigcap_{g \in G} K^g$. Then $K_G \lhd G$ and $|G : K_G| < \infty$ by Lemma 3. If $H < K_G$, then $H \lhd K_G \lhd G$, and hence $H \mathrm{sn} G$, a contradiction. Since H is finite and K_G is infinite, there exists some element $b \in K_G$ and $b \notin H$ such that $\langle a, b\rangle = \langle a\rangle\langle b\rangle \le K$. Since $\langle a\rangle \lhd \langle a, b\rangle$ and $\langle a\rangle$ is non-subnormal, $\langle a, b\rangle$ is non-subnormal, and hence $\langle a, b\rangle$ is conjugate to K. It follows that $\langle b\rangle$ is an infinite cyclic group. Similarly, we have that $\langle a, b^2\rangle = \langle a\rangle\langle b^2\rangle$ is also conjugate to K. It is easily verified that $\langle a, b^2\rangle^b = \langle a, b^2\rangle$ and $b \in N_G(\langle a, b^2\rangle)$. However, $b \notin \langle a, b^2\rangle$, that is $\langle a, b^2\rangle < N_G(\langle a, b^2\rangle)$, which contradicts $N_G(K) = K$. □

Theorem 2. *Let G be an infinite group with $\mu = 2$. If $\mu_{<\infty} = \mu_\infty = 1$, then*

(1) *there exists some normal subgroup $N \lhd G$, such that G/N is a finite non-nilpotent inner-abelian group*;

(2) *G is soluble.*

Proof. Assume that G is an infinite group with $\mu = 2$ and $\mu_{<\infty} = \mu_\infty = 1$. Let H and K be non-subnormal and not conjugate in G, and $H = \langle a\rangle$ be a cyclic group. Since $N_G(H)$ and $N_G(K)$ are also non-subnormal in G, and $\mu_{<\infty} = \mu_\infty = 1$, we have that $N_G(H)$ is conjugate to H, and $N_G(K)$ is conjugate to K.

(1) **Case 1.** $|G : N_G(H)| < \infty$, $|G : N_G(K)| = \infty$.

Now $N_G(H)$ is infinite, that is, H is an infinite cyclic group. Hence $\langle x^p\rangle$ is the normal maximal subgroup of H, for every prime p. Clearly, $\langle x^p\rangle$

is not conjugate to K. Since H has only finitely conjugates, there exist two different maximal normal subgroups $\langle x^p \rangle$ and $\langle x^q \rangle$ of H, such that $\langle x^p \rangle \mathrm{sn} G$, and $\langle x^q \rangle \mathrm{sn} G$, which follows that $H = \langle x^p \rangle \langle x^q \rangle \mathrm{sn} G$, a contradiction.

Case 2. $|G : N_G(H)| = \infty$, $|G : N_G(K)| < \infty$.

If $H < N_G(H) = H^g$, for some $g \in G$, then $H = H_0 \lhd H_1 \lhd \cdots \lhd H_n \lhd \cdots$ is a chain of non-subnormal subgroups of G, where $H_{i+1} = N_G(H_i)$. Moreover, H_i is conjugate to H for all i. Thus H_i is cyclic, and of course is nilpotent. Let the nilpotent class of H be c. Then H_{c+1} is also a nilpotent group with nilpotent class c. According to Lemma 4, $H_c = H_{c+1}$, a contradiction. So $H = N_G(H)$, and hence the maximal subgroup M of H is not conjugate to H, since $M \lhd H \leq N_G(M)$. Clearly, M is not conjugate to K, and hence $M \mathrm{sn} G$. Thus H has only one maximal subgroup according to Lemma 2, which implies that H is a finite cyclic group with order p^m, where p is a prime, and $m \geq 1$. Let $N = K_G = \bigcap_{g \in G} K^g$. Then $N \lhd G$, and G/N is a finite group. Since N is infinite, $N \not\leq H$. Clearly, K/N is non-subnormal in G/N, and G/N has only one class of conjugate non-subnormal subgroups. By Lemma 1, G/N is a non-nilpotent inner-abelian group.

(2) According to Lemma 1, $G/N = K/N \ltimes Q/N$, where K/N is a cyclic group with order p^α, and Q/N is an elementary abelian group with order q^β, where p and q are different primes, and $\alpha, \beta \geq 1$. Let $K/N = \langle gN \rangle$, where $g \notin N$. Then $K = \langle g, N \rangle = \langle g \rangle N$, since $N \lhd K$. Thus $\langle g \rangle$ is non-subnormal in G, since K is non-subnormal in G, and $\langle g \rangle$ is conjugate to H, that is, $H \not\leq N$ by $g \notin N$. It follows that all subgroups of N are subnormal, and N is soluble. Of course, G/N is soluble, and so G is soluble.

Theorem 3. *Let G be an infinite locally finite group with* $\mu = 2$. *Then G is a Baer group*, *and* $\mu_\infty = 2$.

Proof. Let G be an infinite locally finite group with $\mu = 2$, H and K be non-subnormal and not conjugate in G. Assume that G is not locally nilpotent. Then $G = H \ltimes A$, where $H = \langle a \rangle$ is a cyclic p-subgroup for some prime p, A is a nilpotent p'-subgroup and $C_H(A) = \langle g^p \rangle$, according to Lemma 7. Let B be the Baer-radical of G. Then $L = \langle a, b \rangle \not\leq B = A \times \langle a^p \rangle$, for some $1 \neq b \in A$, hence $G = BL$ and $H \leq L$. If $L < N_G(L) = J$, then $H^k \leq L^k = L$ for some $k \in J$, and H^k is a Sylow p-subgroup of L, so there exists some $g \in L$ such that $H^k = H^g$ by Lemma 8, and hence $kg^{-1} \in N_J(H)$, that is, $J = N_J(H)L$. Since $H = N_G(H)$, $H = N_K(H)$, $J = HL = L$, a contradiction. It follows that $L = N_G(L)$, which implies that L is non-subnormal. Since G is locally finite, H and L are finite, thus L is conjugate to K by $H < L$. Similarly, it can be proved that $\langle a, b, g \rangle$ is also non-subnormal in G, for every $1 \neq g \in A$, and hence $\langle a, b, g \rangle$ is conjugate to K, that is, conjugate to L. So $\langle a, b, g \rangle = L$ by $L \leq \langle a, b, g \rangle$, which follows that $g \in L$, for every $1 \neq g \in A$. Thus $G = L$ is finite, a contradiction. Hence G is locally nilpotent, and $\mu_\infty = 2$ by Lemma 6.

According to Lemma 5, without loss of generality, let $H < N_G(H)$. Since $N_G(H)$ is also non-subnormal, $N_G(H)$ is conjugate to H or K. Assume that $N_G(H)$ is conjugate to H. Then there exists some $g \in G$ such that $H < N_G(H) = H^g$. Clearly, G is periodic since G is locally finite, and hence $|g| = n < \infty$. So

$$H < H^g < H^{g^2} < \cdots < H^{g^{n-1}} < H^{g^n} = H,$$

a contradiction. Thus $N_G(H)$ is conjugate to K. Without loss of generality, let $N_G(H) = K$, that is, $H \lhd K$. Clearly, $N_G(K) = K$. Assume that K is finitely generated. Then K is a finite nilpotent group. Since G is an infinite periodic group, there exists some element $g \in G$, $g \notin K$, and $|g| = r$ for some prime r, such that $L = \langle K, g \rangle$ is a finite nilpotent group. Hence $K \lhd L$, which contradicts $N_G(K) = K$. Assume that H is finitely generated. Then $\langle H, k \rangle = K$, for some $k \in K$, $k \notin H$. Otherwise, $H \lhd \langle H, k \rangle \mathrm{sn} G$, and $H \mathrm{sn} G$, a contradiction. So K is finitely generated, a contradiction. That is, H and K are infinitely generated groups, and all cyclic subgroups of G are subnormal. Hence G is a Baer group. □

Remark. Let G be an infinite group with $\mu = 2$. Then Theorem 3 shows that

(1) the length of the conjugacy classes of non-subnormal subgroups of G is infinite;

(2) if H and K are non-subnormal and not conjugate in G, then H, K are infinite. Moreover, $N_G(H) = K$ and $N_G(K) = K$.

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