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ON THE MINIMUM OF ASYMPTOTIC TRANSLATION LENGTHS OF POINT-PUSHING PSEUDO-ANOSOV MAPS

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Abstract

Let S be a closed Riemann surface of genus $p > 1$ with one point x removed. Let $\mathscr{F}$ be the set of mapping classes on S isotopic to the identity on $S \cup \{x\}$. In this paper, we show that for any genus $p > 1$, the minimum $L_{\mathcal{C}}(\mathscr{F})$ of asymptotic translation lengths of all pseudo-Anosov elements of $\mathscr{F}$ satisfies the inequality $\frac{2}{3} \leq L_{\mathcal{C}}(\mathscr{F}) \leq 1$.

1. Introduction and Main Results

Let S be a closed Riemann surface of genus p with n points removed. Assume that $3p - 4 + n > 0$. One can associate to S a curve complex $\mathcal{C}(S)$ which is endowed with a path metric $d_{\mathcal{C}}$. Let $\mathcal{C}_0(S)$ denote the set of vertices

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of $\mathcal{C}(S)$, which can be identified with the set of isotopy classes of simple closed curves on *S*. See Section 2 for the definitions and terminology.

For any $u \in \mathcal{C}_0(S)$, and any pseudo-Anosov map *f* of *S*, we can define $\tau_{\mathcal{C}}(f)$ as

$$\tau_{\mathcal{C}}(f) = \liminf_{m\to\infty} \frac{d_{\mathcal{C}}(u, f^m(u))}{m}. \tag{1.1}$$

It is known that $\tau_{\mathcal{C}}(f)$ does not depend on the choice of $u \in \mathcal{C}_0(S)$ and is called the *asymptotic translation length* for the action of *f* on $\mathcal{C}(S)$. Bowditch [3] proved that $\tau_{\mathcal{C}}(f)$ for each pseudo-Anosov map *f* are rational numbers.

Let $\mathrm{Mod}(S)$ denote the mapping class group of *S*, and let $H \subset \mathrm{Mod}(S)$ be a subgroup. Denote by

$$L_{\mathcal{C}}(H) = \inf\{\tau_{\mathcal{C}}(f) : \text{for all pseudo-Anosov elements in } H\}.$$

By Masur-Minsky [10], there is a positive lower bound for $L_{\mathcal{C}}(H)$ that depends only on (p, n).

For a closed surface *S* of genus $p > 1$, Theorem 1.5 of [5] asserts that

$$L_{\mathcal{C}}(\mathrm{Mod}(S)) < \frac{4\log(2+\sqrt{3})}{p\log\left(p - \frac{1}{2}\right)}.$$

Later, Gadre-Tsai [6] improved the lower and upper bounds for $L_{\mathcal{C}}(\mathrm{Mod}(S))$ as

$$\frac{1}{162(2p-2)^2 + 30(2p-2)} < L_{\mathcal{C}}(\mathrm{Mod}(S)) \le \frac{4}{p^2 + p - 4}.$$

This particularly implies that $L_{\mathcal{C}}(\mathrm{Mod}(S)) \to 0$ as $p \to +\infty$.

The estimations of $L_{\mathcal{C}}(H)$ for certain subgroups *H* of $\mathrm{Mod}(S)$ were also considered in Farb-Leininger-Margalit [5]. Let Γ_0 denote the fundamental group of *S*. For any $k \geq 1$, let Γ_k be the *k*th term of the lower central series for Γ_0. This chain of subgroups forms a filtration. Denote by $\mathscr{N}_k$ the kernel of the natural homomorphism of $\mathrm{Mod}(S)$ onto $\mathrm{Out}(\Gamma/\Gamma_k)$. Then for the sequence of the subgroups $\mathscr{N}_k$, Theorem 6.1 of [5] states that for any *k*, a similar phenomenon emerges. That is, $L_{\mathcal{C}}(\mathscr{N}_k(S)) \to 0$ as $p \to +\infty$.

In this paper, we are mainly concerned with the case in which *S* contains only one puncture *x*. Let $\tilde{S} = S \cup \{x\}$ be equipped with a hyperbolic metric. Then the subgroup $\mathscr{F} \subset \mathrm{Mod}(S)$ that consists of mapping classes projecting to the trivial mapping class on $\tilde{S}$ is highly non-trivial and is isomorphic to the fundamental group $\pi_1(\tilde{S}, x)$. A topological description of this kind of mapping classes is given in [8].

It is well-known (Kra [9]) that $\mathscr{F}$ contains infinitely many pseudo-Anosov elements, and the conjugacy class of a primitive pseudo-Anosov element of $\mathscr{F}$ can be determined by an oriented primitive filling closed geodesic $\tilde{c}$ on $\tilde{S}$. Here a closed geodesic $\tilde{c}$ is said to fill $\tilde{S}$ if all components of $\tilde{S} \backslash \{\tilde{c}\}$ are either (topological) disks or once punctured disks, which is equivalent to that $\tilde{c}$ intersects every simple closed geodesic on $\tilde{S}$.

In contrast to the above estimations for $L_{\mathcal{C}}(H)$ for various subgroups *H* of $\mathrm{Mod}(S)$, in the case where $H = \mathscr{F}$, we can view $L_{\mathcal{C}}(\mathscr{F})$ as a function of (p, n), and see that $L_{\mathcal{C}}(\mathscr{F})$ performs quite differently than $L_{\mathcal{C}}(\mathrm{Mod}(S))$ and $L_{\mathcal{C}}(\mathscr{N}_k(S))$. The main purpose of this paper is to prove the following result.

Theorem 1.1. *For any type* $(p, 1)$ *with* $p > 1$, $\dfrac{2}{3} \leq L_{\mathcal{C}}(\mathscr{F}) \leq 1$.

We may find a filling closed geodesic $\tilde{c}$ on $\tilde{S}$ and a vertex $\tilde{u} \in \mathcal{C}_0(\tilde{S})$ so that $\tilde{u}$ intersects $\tilde{c}$ only once. Let $u \in \mathcal{C}_0(S)$ be the vertex obtained from $\tilde{u}$ by removing *x*. Let $f \in \mathscr{F}$ be a pseudo-Anosov element obtained from pushing x along $\tilde{c}$ (see Theorem 2 of [9]). We know that $\{u, f(u)\}$ forms the boundary of an x-punctured cylinder on S. This means that u and $f(u)$ are disjoint, so that $d_{\mathcal{C}}(u, f(u)) = 1$. By the triangle inequality and the fact that f is a homeomorphism, this gives $d_{\mathcal{C}}(u, f^m(u)) \leq m$ for all $m \geq 1$. It follows from (1.1) that $\tau_{\mathcal{C}}(f) \leq 1$ and thus that $L_{\mathcal{C}}(\mathscr{F}) \leq 1$. The assertion that $L_{\mathcal{C}}(\mathscr{F}) \geq \frac{2}{3}$ follows from the following result.

Theorem 1.2. *Let S be of type $(p, 1)$ with $p > 1$ and let $f \in \mathscr{F}$ be a pseudo-Anosov element. Then there is $u \in \mathcal{C}_0(S)$ such that for any integer m with $|m| \geq 1$, we have*

$$d_{\mathcal{C}}(u, f^m(u)) \geq \begin{cases} |m|, & \text{if } |m| \leq 7, \\ \dfrac{2|m| + 5}{3}, & \text{if } |m| > 7. \end{cases}$$

Remark 1. In [10], Masur-Minsky showed that there is a constant $c = c(p, n)$, $c > 0$, such that $d_{\mathcal{C}}(u, f^m(u)) \geq c|m|$ for all pseudo-Anosov maps f and all $u \in \mathcal{C}_0(S)$. The quantitative estimation for c is, however, largely unknown.

Let $\mathbf{H}$ be a hyperbolic plane and $\varrho : \mathbf{H} \to \tilde{S}$ the universal covering map with a covering group G. Then G is purely hyperbolic. There is an essential hyperbolic element $g \in G$ that corresponds to f (Theorem 2 of [9]). Let $\text{axis}(g) \subset \mathbf{H}$ denote the axis of g; it is the invariant geodesic by the action of g.

In the case where S contains only one puncture x, all vertices u in $\mathcal{C}_0(S)$ are non-preperipheral, in the sense that u is homotopic to a non-trivial simple closed geodesic on $\tilde{S}$ as x is filled in. Thus, for each vertex $u_0 \in \mathcal{C}_0(S)$, there defines a configuration $(\tau_0', \Omega_0', \mathscr{U}_0')$ that corresponds to u_0. See Section 2 for explanations.

For a vertex $\tilde{u} \in \mathcal{C}_0(\tilde{S})$ and a filling geodesic $\tilde{c}$, the geometric intersection number, denoted by $i(\tilde{c}, \tilde{u})$, is defined as the number of intersection points between $\tilde{u}$ and $\tilde{c}$, which is also given by

$$i(\tilde{c}, \tilde{u}) = \min| \tilde{c}' \cap \tilde{u}' |,$$

where $\tilde{c}'$ and $\tilde{u}'$ are in the homotopy classes of $\tilde{c}$ and $\tilde{u}$, respectively. Note that $\tau_{\mathcal{C}}(f)$ does not depend on the choice of $u \in \mathcal{C}_0(S)$. A non-preperipheral vertex $u_0 \in \mathcal{C}_0(S)$ can be selected so that $\Omega_0' \cap \operatorname{axis}(g) \neq \emptyset$ and $i(\varrho(\operatorname{axis}(g)), \tilde{u}_0) \geq 2$.

Outline of the paper. For $m \geq 1$, let u_m be the geodesic homotopic to the image of u_0 under the map f^m. Suppose that

$$[u_0, v_1, ..., v_s, u_m] \tag{1.2}$$

is an arbitrary geodesic path in the 1-skeleton of $\mathcal{C}(S)$ that connects u_0 and u_m with a minimum number of sides. Then all v_j, $1 \leq j \leq s$, are non-preperipheral, which allows us to obtain the configurations $(\tau_j, \Omega_j, \mathscr{U}_j)$ determined by the vertices v_j.

Observe that the sequence $\mathbf{H}\backslash\Delta'_j$ (see Figure 2 and (3.2) for the definition of Δ'_j) monotonically moves down towards the attracting fixed point of g. The sequence Ω_j tends to move out of Δ'_m. We show that if v_j and v_{j+1} are disjoint, then Ω_j and Ω_{j+1} are either adjacent or $\Omega_j \cap \Omega_{j+1} \neq \emptyset$. So the movement of Ω_j is not too fast. This means that a sufficient amount of Ω_j is needed to get out of Δ'_m. Careful analysis shows that for any $m \geq 7$, the distance $d_{\mathcal{C}}(u_0, u_m)$ is greater than or equal to $(2m+5)/3$, and if $0 < m < 7$, then $d_{\mathcal{C}}(u_0, u_m) \geq m$. It follows that $\frac{2}{3} \leq L_{\mathcal{C}}(\mathscr{F}) \leq 1$. If m is negative and $m \leq -7$, the proof is similar.

2. Curve Complex and Tessellations in Hyperbolic Plane

Let S be a hyperbolic surface which is of type (p, n) with $3p - 4 + n > 0$ and $n \geq 1$. Let x be a puncture of S and let $\tilde{S} = S \cup \{x\}$ be also equipped with a hyperbolic metric.

Due to Harvey [7], one can define the curve complex $\mathcal{C}(S)$ of dimension $3p - 4 + n$ as the following simplicial complex: vertices of $\mathcal{C}(S)$ are isotopy classes of simple closed curves, and k-dimensional simplicies of $\mathcal{C}(S)$ are collections of $(k+1)$-tuples $\{u_0, u_1, ..., u_k\}$ of disjoint vertices on S, where two vertices u_i and u_j are disjoint if there exist disjoint representatives of u_i and u_j. Then $\mathcal{C}(S)$ is of finite dimensional. Let $\mathcal{C}_k(S)$ denote the k-skeleton of $\mathcal{C}(S)$. We then introduce a metric $d_{\mathcal{C}}$ on $\mathcal{C}(S)$, called the *path metric*, in the following way. First we make each simplex Euclidean with side length one, then for any vertices $u, v \in \mathcal{C}_0(S)$, we declare the distance $d_{\mathcal{C}}(u, v)$ between u and v to be the smallest number of edges in $\mathcal{C}_1(S)$ connecting u and v. It is well-known that $\mathcal{C}(S)$ is connected and is δ-hyperbolic in the sense of Gromov (Masur-Minsky [10]).

The curve complex $\mathcal{C}(\tilde{S})$ is similarly defined. Every vertex in $\mathcal{C}_0(S)$ or $\mathcal{C}_0(\tilde{S})$ can be identified with a simple closed geodesic. Let $\hat{\mathcal{C}}(S)$ be the subcomplex of $\mathcal{C}(S)$ consisting of non-preperipheral vertices. Thus, there defines a fibration $\hat{\mathcal{C}}(S) \to \mathcal{C}(\tilde{S})$ by forgetting the puncture x. According to Birman-Series [4], the union of all simple closed geodesics on $\tilde{S}$ is not all of $\tilde{S}$. Whence we may choose a point x that misses every simple closed geodesic on $\tilde{S}$, which means that a vertex in $\mathcal{C}(\tilde{S})$ can also be regarded as a vertex on $\hat{\mathcal{C}}(S)$ (by simply removing the point x). We see that the fibration $\hat{\mathcal{C}}(S) \to \mathcal{C}(\tilde{S})$ admits a global section.

Let G be the group of covering transformations of the universal covering map $\varrho : \mathbf{H} \to \tilde{S}$. The x-pointed mapping class group Mod_S^x, which is defined as a group that consists of mapping classes fixing x, is a subgroup of the ordinary mapping class group $\mathrm{Mod}(S)$ with finite index n. It is well-known (Theorem 4.1 and Theorem 4.2 of Birman [2]) that there exists an exact sequence

$$0 \to \pi_1(\tilde{S}, x) \to \mathrm{Mod}_S^x \to \mathrm{Mod}(\tilde{S}) \to 0, \tag{2.1}$$

which defines an injective map of $\pi_1(\tilde{S}, x)$ into Mod_S^x. Note that an isomorphism between G and $\pi_1(\tilde{S}, x)$ is obtained by choosing a lift $\hat{x}$ of x in $\mathbf{H}$ to serve as a base point. As such, we obtain an injective map $\psi : G \to \mathrm{Mod}_S^x$.

Following Bers [1], we denote by $\mathscr{A} = \{h(\hat{x}) : h \in G\} \subset \mathbf{H}$. Then $\mathscr{A}$ is a discrete subset of $\mathbf{H}$ invariant under the action of G. Let $\dot{G}$ be the covering group of a universal covering map $\varrho' : \mathbf{H} \to S$. Clearly, $\mathbf{H}/\dot{G} \cong S \cong (\mathbf{H}/G) \backslash \{x\}$, and there exists an exact sequence

$$1 \to \Gamma \to \dot{G} \to G \to 1,$$

where Γ is the covering group of a universal covering map $v : \mathbf{H} \to \mathbf{H}\backslash\mathscr{A}$.

Let $Q(G)$ (resp. $Q(\dot{G})$) be the group of quasiconformal automorphisms w of $\mathbf{H}$ satisfying $wGw^{-1} = G$ (resp. $w\dot{G}w^{-1} = \dot{G}$). Two elements $w_1, w_2 \in Q(G)$ (or in $Q(\dot{G})$) are declared to be equivalent (write $w_1 \sim w_2$) if $w_1 = w_2$ on $\partial\mathbf{H} = \mathbf{S}^1$. Let $Q_0(\dot{G})$ denote the subgroup of $Q(\dot{G})$ consisting of maps projecting (under ϱ') to maps on S leaving the puncture x fixed. For any $w \in Q(G)$, there is a map $w_0 \in Q(G)$ with $[w] = [w_0]$ such that $w_0(\mathscr{A}) = \mathscr{A}$. Thus w_0 defines a map on $\mathbf{H}\backslash\mathscr{A}$, and hence w_0 can be lifted (through ϱ') to a map $\omega_0 \in Q_0(\dot{G})$. By Theorem 10 of [1], the map $\varphi^* : Q(G)/\sim \to$

Mod_S^x defined by sending $[w]$ to the mapping class represented by the projection of ω_0 under ϱ' is an isomorphism. For simplicity we denote by $[w]^* = \varphi^*([w])$ for a $[w] \in Q(G)/\sim$. It is known that G can be regarded as a normal subgroup of $Q(G)/\sim$ so that the restriction $\varphi^*|G$ is exactly the injective map $\psi : G \to \mathrm{Mod}_S^x$. In other words, we have $\varphi^*(G) = \psi(G) = \mathscr{F}$. Let $h^* \in \mathscr{F} \subset \mathrm{Mod}_S^x$ denote the mapping class $\varphi(h) = \psi(h)$ for an $h \in G$.

Fix $\tilde{\varepsilon} \in \mathcal{C}_0(\tilde{S})$. Let $\{\varrho^{-1}(\tilde{\varepsilon})\}$ be the set of geodesics $\hat{\varepsilon}$ in $\mathbf{H}$ with $\varrho(\hat{\varepsilon}) = \tilde{\varepsilon}$. As $\tilde{\varepsilon}$ is simple, all geodesics in $\{\varrho^{-1}(\tilde{\varepsilon})\}$ are mutually disjoint. By Theorem 2 of Kra [9], there is a bijection Φ of $\{\varrho^{-1}(\tilde{\varepsilon})\}$ onto the set $\mathscr{P}$ of x-punctured cylinders on S whose geodesic boundary components project to $\tilde{\varepsilon}$. Two such cylinders $C, C' \in \mathscr{P}$ are called *equivalent* (denoted by $C \sim C'$) if they share one boundary component.

It is clear that $\{\varrho^{-1}(\tilde{\varepsilon})\}$ gives rise to a partition of $\mathbf{H}$. Let $\mathscr{R}_{\tilde{\varepsilon}}$ denote the set of components of $\mathbf{H}\backslash\{\varrho^{-1}(\tilde{\varepsilon})\}$. Let $t_{\tilde{\varepsilon}}$ be the positive Dehn twist along $\tilde{\varepsilon}$, which is supported in a small neighborhood $\tilde{\mathscr{N}}$ of $\tilde{\varepsilon}$. Let $\mathscr{N} \subset \mathbf{H}$ be the union of all thin neighborhoods of $\hat{\varepsilon} \in \{\varrho^{-1}(\tilde{\varepsilon})\}$ so that $\varrho(\mathscr{N}) = \tilde{\mathscr{N}}$. For every $\Omega \in \mathscr{R}_{\tilde{\varepsilon}}$, there is a lift $\tau : \mathbf{H} \to \mathbf{H}$ of $t_{\tilde{\varepsilon}}$ so that the restriction $\tau|_{\Omega\backslash\mathscr{N}} = \mathrm{id}$. It is easy to see that $\tau \in Q(G)\backslash G$ and thus $[\tau] \in (Q(G)/\sim)\backslash G$. Let $F_{\tilde{\varepsilon}} \subset \hat{\mathcal{C}}_0(S)$ denote the fiber over $\tilde{\varepsilon}$ that consists of all $u \in \hat{\mathcal{C}}_0(S)$ for which $\tilde{u} = \tilde{\varepsilon}$, where $\tilde{u}$ is homotopic to u if u is viewed as a curve on $\tilde{S}$. By Lemma 3.2 of [13], $[\tau]^*$ is represented by the positive Dehn twist t_u along a non-preperipheral geodesic u for a $u \in F_{\tilde{\varepsilon}}$. More precisely, the argument of the lemma yields that $\Phi(\hat{c}) \sim \Phi(\hat{c}')$ for any two boundary components $\hat{c}, \hat{c}' \in \partial\Omega \subset \{\varrho^{-1}(\tilde{\varepsilon})\}$. The components of $\partial\Omega$ are one-to-one correspondent with the elements in the equivalence class of $\Phi(\hat{c})$ in $\mathscr{P}$. Thereby we obtain for each $\tilde{\varepsilon} \in \mathcal{C}_0(\tilde{S})$ a well-defined surjective map

$$\chi_{\tilde{\varepsilon}} : \mathscr{R}_{\tilde{\varepsilon}} \to F_{\tilde{\varepsilon}} \tag{2.2}$$

which sends Ω to u. $\chi_{\tilde{\varepsilon}}$ satisfies the invariance property: For every $h \in G$, we have $\tau_{h(\Omega)} = h(g^{-1}\tau)h^{-1}$. Hence

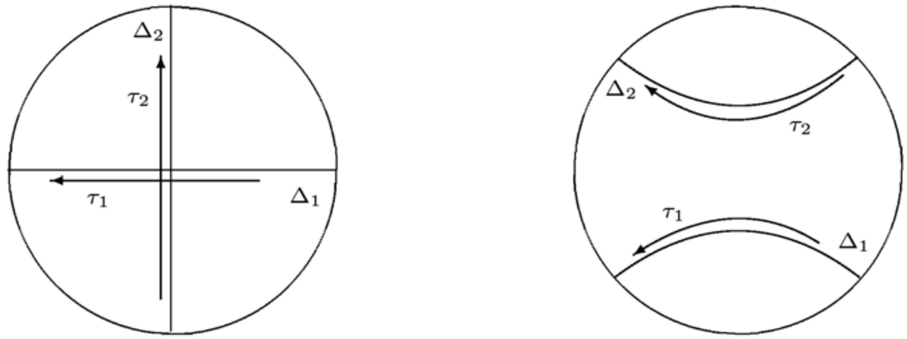
$$[\tau_{h(\Omega)}]^* = [h(g^{-1}\tau)h^{-1}]^* = h^*[\tau_\Omega]^*(h^{*-1}) = t_{h^*(u)}.$$

Let $\Omega \in \mathscr{R}_{\tilde{\varepsilon}}$ be such that $\chi_{\tilde{\varepsilon}}(\Omega) = u$ for a given $u \in F_{\tilde{\varepsilon}}$. Observe that the complement of the closure of Ω is a disjoint union of half planes. Each such half plane Δ includes infinitely many geodesics in $\{\varrho^{-1}(\tilde{\varepsilon})\}$, and no geodesics in $\{\varrho^{-1}(\tilde{\varepsilon})\}$ are contained in Ω. Thus, there are infinitely many half planes contained in Δ. Let $\mathscr{U}$ be the collection of all such half planes. Obviously $\mathscr{U}$ is a partially ordered set defined by inclusion. Maximal elements of $\mathscr{U}$ are called *first order elements* (Δ is one of them), elements of $\mathscr{U}$ that are included in a maximal element but are not included in any other elements of $\mathscr{U}$ are called *second order elements*, and so on. We call the triple $(\tau, \Omega, \mathscr{U})$ the configuration corresponding to u.

For any two vertices $u_1, u_2 \in \hat{\mathcal{C}}_0(S)$, let $(\tau_1, \Omega_1, \mathscr{U}_1)$ and $(\tau_2, \Omega_2, \mathscr{U}_2)$ be the configurations corresponding to u_1 and u_2, respectively. That is, $u_1 = \chi_{\tilde{\varepsilon}}(\Omega_1)$ and $u_2 = \chi_{\tilde{\varepsilon}}(\Omega_2)$.

Lemma 2.1. *Assume that there are maximal elements* $\Delta_1 \in \mathscr{U}_1$ *and* $\Delta_2 \in \mathscr{U}_2$ *such that* $\Delta_1 \cap \Delta_2 \neq \varnothing$ *and that* Δ_1 *is neither contained in* Δ_2 *nor contains* Δ_2. *Then* $d_{\mathcal{C}}(u_1, u_2) \geq 2$.

Proof. We are left with two cases drawn in Figures 1(a) and (b). If Figure 1(a) occurs, then since $\varrho : \mathbf{H} \to \tilde{S}$ is a local homeomorphism, $\tilde{u}_1$ intersects $\tilde{u}_2$, which leads to that u_1 intersects u_2.



(a) (b)

Figure 1

If Figure 1(b) occurs, then by considering the iterations $\tau_1^n\tau_2^m$ and $\tau_2^m\tau_1^n$ on the unit circle $\mathbf{S}^1$ for large n and m, from Lemma 4 of [12], we conclude that $\tau_2^m\tau_1^n \neq \tau_1^n\tau_2^m$. Thus $t_{u_2}^m \circ t_{u_1}^n \neq t_{u_1}^n \circ t_{u_2}^m$. It follows that u_1 intersects u_2. $\square$

In particular, in the case where $\tilde{u}_1 = \tilde{u}_2 = \tilde{\varepsilon}$, i.e., u_1, u_2 lie on the fiber $F_{\tilde{\varepsilon}}$, then $\Omega_1, \Omega_2 \in \mathscr{R}_{\tilde{\varepsilon}}$. If $\overline{\Omega}_1$ is disjoint from $\overline{\Omega}_2$, Lemma 2.1 asserts that u_1 intersects u_2. Now consider the case where Ω_1 and $\Omega_2 \in \mathscr{R}_{\tilde{\varepsilon}}$ are adjacent; that is, Ω_1 and Ω_2 share a common geodesic in $\mathbf{H}$.

Lemma 2.2. *Let* $\Omega_1, \Omega_2 \in \mathscr{R}_{\tilde{\varepsilon}}$. *The following are equivalent*:

(i) Ω_1 *and* Ω_2 *are adjacent,*

(ii) $d_{\mathcal{C}}(u_1, u_2) = 1$, *and*

(iii) $\{u_1, u_2\}$ *are boundary components of an x-punctured cylinder on S.*

Proof. Suppose that Ω_1 and Ω_2 are adjacent. Then $\Omega_2 \subset \Delta_1$ for an element $\Delta_1 \in \mathscr{U}_1$ and $\Omega_2 = \Delta_1 \setminus \{$all second order elements of $\mathscr{U}_1$ in $\Delta_1\}$. Let

$e = \overline{\Omega}_1 \cap \overline{\Omega}_2$ denote the common boundary geodesic in **H**. As usual, let τ_i, $i = 1, 2,$ be the lifts of $t_{\tilde{\varepsilon}}$ that is defined by Ω_i. Let $u_1 = \chi_{\tilde{\varepsilon}}(\Omega_1)$ and $u_2 = \chi_{\tilde{\varepsilon}}(\Omega_2)$. Let $\mathscr{N}_e$ be the 1-neighborhood of e. Then $\mathscr{N}_e$ is a crescent neighborhood that touches $\mathbf{S}^1$ at two points $\{y, z\}$. Let $h \in G$ be the primitive simple hyperbolic element that keeps e (and hence also $\mathscr{N}_e$) invariant and has the same orientation as τ_1. It is easy to see that $\{y, z\}$ are fixed points of h and that $\mathscr{N}_e / \langle h \rangle$ is a cylinder with central geodesic $\tilde{\varepsilon}$. From Theorem 2 of [9, 11], h^* is represented by the spin $t_{u_1} \circ t_{u_0}^{-1}$, where we note that u_1 and u_0 are boundary components of an x-punctured cylinder on S.

On the other hand, by construction, $h^{-1}\tau_1$ leaves the identity on Ω_2. Clearly, Ω_2 is the maximal region so that the restriction $h^{-1}\tau_1|_{\Omega_2 \setminus \mathscr{N}}$ is the identity. It follows that $h^{-1}\tau_1 = \tau_2$. As such, by the construction of $\chi_{\tilde{\varepsilon}}$, $[h^{-1}\tau_1]^* = [\tau_2]^* = t_{u_2}$ and $[\tau_1]^* = t_{u_1}$. But

$$[h^{-1}\tau_1]^* = (h^*)^{-1} \circ [\tau_1]^* = t_{u_0} \circ t_{u_1}^{-1} \circ t_{u_1} = t_{u_0}.$$

It follows that $t_{u_0} = t_{u_2}$ and thus $u_0 = u_2$. Since u_1 and u_0 are boundary components of an x-punctured cylinder on S, u_1 and u_0 are disjoint. So u_1 and u_2 are disjoint. This particularly implies that $d_{\mathcal{C}}(\chi_{\tilde{\varepsilon}}(\Omega_1), \chi_{\tilde{\varepsilon}}(\Omega_2)) = 1$.

Conversely, suppose that $d_{\mathcal{C}}(u_1, u_2) = 1$. Since $\tilde{u}_1 = \tilde{u}_2 = \tilde{\varepsilon}$, $\{u_1, u_2\}$ forms an x-punctured cylinder on S. It follows from Theorem 2 of [9, 11] that there is a simple hyperbolic element $h \in G$ such that $h^* = t_{u_1} \circ t_{u_2}^{-1}$. That is, $\tau_2 = h^{-1}\tau_1$, which implies that Ω_1 and Ω_2 are adjacent. Finally, the fact that (ii) and (iii) are equivalent is obvious. □

It follows from Lemma 2.2 and Lemma 2.1 that the map (2.2) is also injective. Hence we have established the following:

Lemma 2.3. *The map* $\chi_{\tilde{\varepsilon}}$ *defined as* (2.2) *is a bijection which satisfies the equivariance condition* $\chi_{\tilde{\varepsilon}}(g(\Omega)) = g^*(\chi_{\tilde{\varepsilon}}(\Omega))$ *for any* $\tilde{\varepsilon} \in \mathcal{C}_0(\tilde{S})$, $\Omega \in \mathscr{R}_{\tilde{\varepsilon}}$, *and* $g \in G$.

Suppose now that $\Omega_1, \Omega_2 \in \mathscr{R}_{\tilde{\varepsilon}}$ and $\Omega_1 \neq \Omega_2$. Then Ω_1 is disjoint from Ω_2. There are two cases: either Ω_1 is adjacent to Ω_2, or $\overline{\Omega}_1$ is disjoint from $\overline{\Omega}_2$. If Ω_1 is adjacent to Ω_2, then by Lemma 2.2, $d_{\mathcal{C}}(u_1, u_2) = 1$. If $\overline{\Omega}_1$ is disjoint from $\overline{\Omega}_2$, by Lemma 2.1, we have $d_{\mathcal{C}}(u_1, u_2) \geq 2$.

Remark 2. In [17], we further discussed the case where $d_{\mathcal{C}}(u_1, u_2) \geq 2$, and give a characterization for the two geodesics $u_1 = \chi_{\tilde{\varepsilon}}(\Omega_1)$, $u_2 = \chi_{\tilde{\varepsilon}}(\Omega_2)$ to satisfy the condition $d_{\mathcal{C}}(u_1, u_2) = 2$.

In the case where $\tilde{u}_1 \neq \tilde{u}_2$, we have:

Lemma 2.4. *Let* $u_1, u_2 \in \mathcal{C}_0(S)$ *be such that* $\tilde{u}_1 \neq \tilde{u}_2$. *The following statements are equivalent*:

(i) $d_{\mathcal{C}}(u_1, u_2) = 1$,

(ii) $\Omega_1 \cap \Omega_2 \neq \varnothing$ *and* $\{\varrho^{-1}(\tilde{u}_1)\}$ *is disjoint from* $\{\varrho^{-1}(\tilde{u}_2)\}$, *and*

(iii) *for any maximal elements* $\Delta_1 \in \mathscr{U}_1$ *and* $\Delta_2 \in \mathscr{U}_2$, *either* Δ_1 *and* Δ_2 *are disjoint, or* Δ_1 *and* Δ_2 *are nested* (*that is, either* $\Delta_1 \subset \Delta_2$, *or* $\Delta_2 \subset \Delta_1$).

Proof. If u_1, u_2 are disjoint, then $\tilde{u}_1, \tilde{u}_2$ are also disjoint. Hence $\{\varrho^{-1}(\tilde{u}_1)\}$ and $\{\varrho^{-1}(\tilde{u}_2)\}$ are disjoint. The fact that $\Omega_1 \cap \Omega_2 \neq \varnothing$ follows from Lemma 2.1. This shows that (i) implies (ii).

To show that (ii) implies (iii), we notice that τ_1 and τ_2 are the lifts of $t_{\tilde{u}_1}$ and $t_{\tilde{u}_1}$. τ_1 and τ_2 determine the configurations $(\tau_1, \Omega_1, \mathscr{U}_1)$ and

$(\tau_2, \Omega_2, \mathscr{U}_2)$. Since $\{\varrho^{-1}(\tilde{u}_1)\}$ and $\{\varrho^{-1}(\tilde{u}_2)\}$ are disjoint and $\Omega_1 \cap \Omega_2 \neq \emptyset$, (iii) is satisfied.

Finally, if (iii) holds, then $\Omega_1 \cap \Omega_2 \neq \emptyset$ and $\tilde{u}_1$ and $\tilde{u}_2$ are disjoint. If u_1 intersects u_2, then we are in the situation of Figure 1(b), and we must have $\Omega_1 \cap \Omega_2 \neq \emptyset$. This leads to a contradiction. □

Remark 3. If a maximal element $\Delta_1 \in \mathscr{U}_1$ contains a maximal element of $\mathscr{U}_2$, then Δ_1 contains infinitely many maximal elements of $\mathscr{U}_2$; but if $\Delta_1 \in \mathscr{U}_1$ is contained in a maximal element Δ_2 of $\mathscr{U}_2$, then such a Δ_2 is unique. The same is true for maximal elements of $\mathscr{U}_2$.

Throughout the rest of the paper we assume that S is a closed hyperbolic Riemann surface minus one point x. In this case, we have $\mathrm{Mod}_S^x = \mathrm{Mod}(S)$ and $\mathcal{C}_0(S) = \hat{\mathcal{C}}_0(S)$; that is, every vertex $u \in \mathcal{C}_0(S)$ is non-preperipheral.

3. Partitions and Regions in Hyperbolic Plane Determined by Vertices

Let $f \in \mathscr{F}$ be a pseudo-Anosov element. By Theorem 2 of [9], there is $g \in G$ such that $g^* = f$ and g is an essential hyperbolic element, which means that the projection $\tilde{c} := \varrho(\mathrm{axis}(g))$ is an oriented filling closed geodesic on $\tilde{S}$.

Let $\tilde{u}_0 \in \mathcal{C}_0(\tilde{S})$. Then $i(\tilde{u}_0, \tilde{c}) \geq 1$. Choose $\Omega_0' \in \mathscr{R}_{\tilde{u}_0}$ so that $\Omega_0' \cap \mathrm{axis}(g) \neq \emptyset$. Obviously, Ω_0' determines a configuration $(\tau_0', \Omega_0', \mathscr{U}_0')$ that corresponds to a vertex $\chi_{\tilde{u}_0}(\Omega_0') = u_0 \in F_{\tilde{u}_0} \subset \mathcal{C}_0(S)$. Notice that $\tilde{c} \subset \tilde{S}$ is a filling geodesic that intersects $\tilde{u}_0$. Thus $\mathrm{axis}(g)$ crosses infinitely many geodesics in $\{\varrho^{-1}(\tilde{u}_0)\}$. This particularly implies that $\mathrm{axis}(g)$ cannot be completely included in Ω_0'.

Since $\tilde{u}_0$ is simple, all geodesics in $\{\varrho^{-1}(\tilde{u}_0)\}$ are mutually disjoint. We conclude that there are maximal elements $\Delta_0, \Delta'_0 \in \mathscr{U}'_0$, which are disjoint from each other, such that $\text{axis}(g)$ crosses both Δ_0 and Δ'_0 and that $\Omega'_0 \subset \mathbf{H} \backslash (\Delta_0 \cup \Delta'_0)$. We may assume that Δ_0 and Δ'_0 cover the attracting and repelling fixed points $\{A, B\}$ of g, respectively. Δ_0 and Δ'_0 are shown in Figure 2.

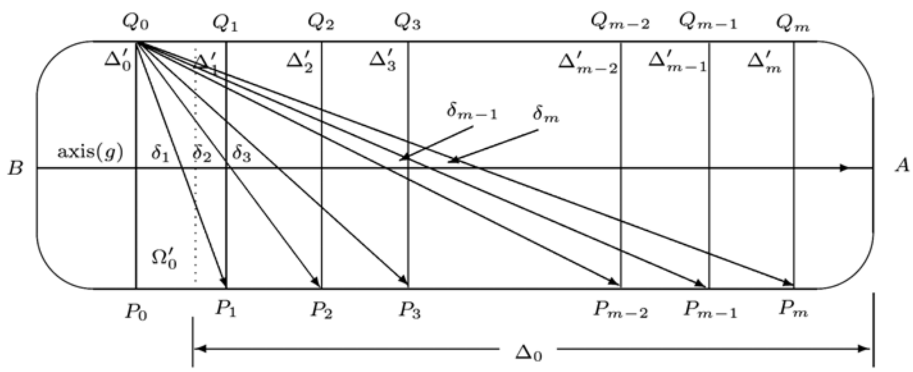


Figure 2

For each $k \geq 1$, let u_k denote the geodesic homotopic to the image of u_0 under the map f^k. Then u_k is a non-preperipheral geodesic and in particular,

$$(\tau'_k, \Omega'_k, \mathscr{U}'_k) := (g^k \tau'_0 g^{-k}, g^k(\Omega'_0), g^k(\mathscr{U}'_0)) \tag{3.1}$$

is the configuration corresponding to u_k. This tells us that

$$\Delta'_k := g^k(\Delta'_0) \tag{3.2}$$

is a maximal element of $\mathscr{U}'_k$ that covers the repelling fixed point B of g. All the half planes $\{\Delta'_k\}$ are also shown in Figure 2.

For any half plane $\Delta \subset \mathbf{H}$, denote by $\partial\Delta$ the boundary geodesic of Δ in $\mathbf{H}$. It is obvious that for $k \geq 0$,

$$g(\partial\Delta'_k) = \partial\Delta'_{k+1}. \tag{3.3}$$

For simplicity, we also write $\{P_k, Q_k\} = \mathbf{S}^1 \cap \partial\Delta'_k$. Observe that $\mathbf{S}^1 \backslash \{A, B\}$ has two components $\mathcal{L}$ and $\mathcal{R}$, where $\mathcal{L}$ is the component containing all labeled points Q_i and $\mathcal{R}$ is the component containing all labeled points P_i.

For any two points $X, X' \in \mathcal{L}$, denote by (XX') the subarc of $\mathcal{L}$ connecting X and X'. Likewise, let (YY') denote the subarc of $\mathcal{R}$ connecting two points $Y, Y' \in \mathcal{R}$. Points lying on $\mathcal{L}$ or on $\mathcal{R}$ can be ordered as follows: we declare $X < X'$ if and only if $(BX) \subset (BX') \subset \mathcal{L}$. Likewise, we say $Y < Y'$ if and only if $(BY) \subset (BY') \subset \mathcal{R}$.

Some basic properties for the labeled points P_k and Q_k are included in the following lemma.

Lemma 3.1. *We have*:

(i) *all* P_k *and* Q_k *are hyperbolic fixed points of G satisfying* $P_1 < P_2 < \cdots < P_m < \cdots$ *and* $Q_1 < Q_2 < \cdots < Q_m < \cdots$,

(ii) *for all* $k \geq 0$, $g(Q_k) = Q_{k+1}$ *and* $g(P_k) = P_{k+1}$, *and more precisely*,

(iii) *for all* $k \geq 0$, $g(Q_k Q_{k+1}) = (Q_{k+1} Q_{k+2})$ *and* $g(P_k P_{k+1}) = (P_{k+1} P_{k+2})$.

Proof. Notice that $\{P_k, Q_k\} = \partial\Delta'_k \cap \mathbf{S}^1$ and $\Delta'_k \in \mathscr{U}_k$ are maximal elements. We deduce that $\partial\Delta'_k$ are axes of simple hyperbolic elements of G. It turns out that P_k and Q_k are hyperbolic fixed points of G. Since $\varrho(\partial\Delta'_k) = \tilde{u}_0$ for all $k \geq 0$, $\{\partial\Delta'_k : k \geq 0\} \subset \{\varrho^{-1}(\tilde{u}_0)\}$. Hence $\partial\Delta'_{k+1}$ is disjoint from $\partial\Delta'_k$ for all $k \geq 0$. (i)-(iii) follow immediately from (3.2) and (3.3). □

Lemma 3.2. *Let* $\hat{u} \subset \mathbf{H}$ *be a geodesic that divides* $\mathbf{H}$ *into two half planes* Δ *and* Δ'. *Suppose that* $\varrho(\hat{u}) \subset \tilde{S}$ *is a simple closed geodesic and* Δ *is disjoint from* $\operatorname{axis}(g)$ *and covers* Q_k *for some k. Then* Δ *cannot cover any other labeled point* Q_i.

Proof. Since $\varrho(\hat{u}) \in \mathcal{C}_0(\widetilde{S})$ is simple, for any $h \in G$, $h(\partial\Delta) \bigcap \partial\Delta = \varnothing$. In particular, $g(\partial\Delta) \bigcap \partial\Delta = \varnothing$. Consider the action of g on $\mathbf{H}$. For those Δ disjoint from $\operatorname{axis}(g)$, we have $g(\Delta) \bigcap \Delta = \varnothing$. If Δ covers Q_k and Q_{k+1}, and is disjoint from $\operatorname{axis}(g)$, then by Lemma 3.1(iii), $g(\Delta)$ covers Q_{k+1} and Q_{k+2}, which implies that $g(\Delta) \bigcap \Delta \neq \varnothing$, contradicting that $g(\Delta) \bigcap \Delta = \varnothing$. □

Let $\delta_0 = \alpha_0$ be the angle between $\partial\Delta_0'$ and $\operatorname{axis}(g)$. Unless otherwise stated, throughout the paper the angle between $\operatorname{axis}(g)$ and a geodesic l intersecting $\operatorname{axis}(g)$ is defined as the angle through which l must be rotated (in counterclockwise direction) to make it coincide with $\operatorname{axis}(g)$. For $j \geq 1$, we let δ_j be the angle between $\operatorname{axis}(g)$ and the geodesic joining from Q_0 to P_j. See Figure 2. We see that

$$\cdots < \delta_m < \delta_{m-1} < \cdots < \delta_1 < \delta_0.$$

Since g can be regarded as a Möbius transformation of $\mathbf{H}$ which keeps $\operatorname{axis}(g)$ invariant, δ_j is also the angle between $\operatorname{axis}(g)$ and the geodesic joining from Q_k to P_{j+k} for all $k \geq 0$.

More generally, δ_j are invariant under Möbius transformations on $\mathbf{H}$. This in turn implies that for every $h \in G$, δ_j is also the angle between $h(\operatorname{axis}(g))$ and the geodesic connecting $h(Q_0)$ and $h(P_j)$.

To each geodesic $\hat{u}$ intersecting $\operatorname{axis}(g)$, there associates an angle $\delta_{\hat{u}}$ which is defined as follows. Write $\{X_{\hat{u}}, Y_{\hat{u}}\} = \hat{u} \bigcap \mathbf{S}^1$, where $X_{\hat{u}} \in \mathcal{L}$ and $Y_{\hat{u}} \in \mathcal{R}$. Let $\hat{u}'$ be the geodesic connecting $Y_{\hat{u}}$ and $g^{-1}(X_{\hat{u}})$. Then we define $\delta_{\hat{u}}$ as the angle between $\hat{u}'$ and $\operatorname{axis}(g)$.

Lemma 3.3. *Let $\hat{v} \subset \mathbf{H}$ be a geodesic intersecting axis(g). Let $\alpha_{\hat{v}}$ denote the angle between $\hat{v}$ and* $\operatorname{axis}(g)$. *Suppose that $\alpha_{\hat{v}} < \delta_{\hat{u}}$. Then $\{g^i(\hat{v}) : i \geq 0\}$ intersects $\hat{u}$.*

Proof. Suppose that $g^i(\hat{v})$ are disjoint from $\hat{u}$ for all i. Then $g^i(\hat{u})$ are disjoint from $\hat{v}$ for all i. Let p be the integer such that $\hat{v}$ lies in between $g^p(\hat{u})$ and $g^{p+1}(\hat{u})$. See Figure 3.

It is obvious that $\alpha_{\hat{v}}$ is no smaller than the angle α between $\text{axis}(g)$ and the geodesic connecting $g^p(X_{\hat{u}})$ and $g^{p+1}(Y_{\hat{u}})$. Note that g keeps $\text{axis}(g)$ invariant. α is also the angle between $\text{axis}(g)$ and the geodesic connecting $g^{-1}(X_{\hat{u}})$ and $Y_{\hat{u}}$ which is, by the definition, is equal to $\delta_{\hat{u}}$. We thus conclude that $\alpha = \delta_{\hat{u}}$, and so $\alpha_{\hat{v}} \geq \delta_{\hat{u}}$, which contradicts the hypothesis. □

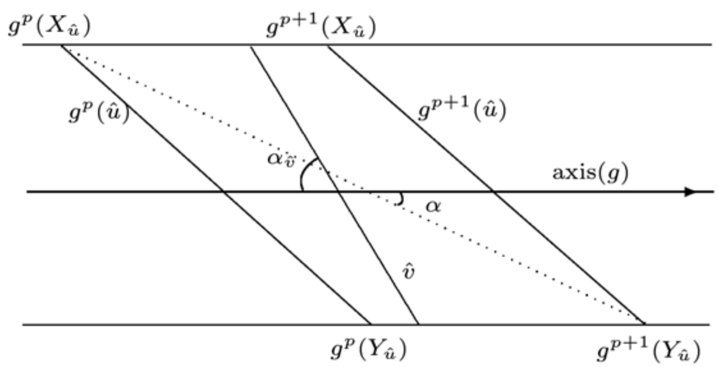


Figure 3

The following lemma will be used frequently in the proof of Theorem 1.2.

Lemma 3.4. *Let* $\hat{u}, \hat{v}$ *be geodesics in* **H** *intersecting* $\text{axis}(g)$. *Let* $\alpha_{\hat{u}}$ *and* $\alpha_{\hat{v}}$ *be the angles between* $\text{axis}(g)$ *and* $\hat{u}$ *and between* $\text{axis}(g)$ *and* $\hat{v}$, *respectively. Assume that* $\delta_q \leq \alpha_{\hat{u}} < \delta_{q-1}$ *for some* $q \geq 1$. *Then* $\{g^i(\hat{v}) : i \geq 0\}$ *intersects* $\hat{u}$ *whenever* $\alpha_{\hat{v}} < \delta_{q+1}$.

Proof. The assumption that $\delta_q \leq \alpha_{\hat{u}} < \delta_{q-1}$ and $\delta_q \geq \delta_{q+1}$ implies that $\delta_{\hat{u}} \geq \delta_{q+1}$. Since $\alpha_{\hat{v}} < \delta_{q+1}$, we have $\alpha_{\hat{v}} < \delta_{\hat{u}}$ and hence by Lemma 3.3, we conclude that $\{g^i(\hat{v}) : i \geq 0\}$ intersects the geodesic $\hat{u}$. □

4. Classification of Vertices of $\mathcal{C}(S)$ in Terms of Essential Hyperbolic Elements of G

For each $v \in \mathcal{C}_0(S)$, $\tilde{v} \in \mathcal{C}_0(\tilde{S})$ is non-trivial. Let $(\tau_v, \Omega_v, \mathscr{U}_v)$ denote the corresponding configuration. By the definition, Ω_v is a component of $\mathbf{H} \backslash \{\varrho^{-1}(\tilde{v})\}$. As we saw before, $\text{axis}(g)$ cannot be completely contained in Ω_v. There are two possibilities:

(1) $\text{axis}(g)$ is contained in a maximal element Δ_v of $\mathscr{U}_v$, or

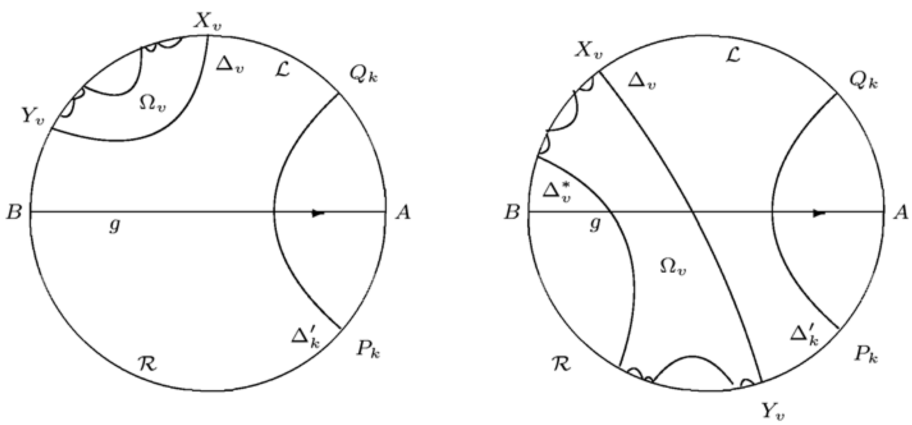
(2) there is a pair (Δ_v, Δ_v^*) of maximal elements of $\mathscr{U}_v$ such that $\text{axis}(g)$ intersects $\partial\Delta_v$ and $\partial\Delta_v^*$, where Δ_v and Δ_v^* are the maximal elements of $\mathscr{U}_v$ that cover the attracting and repelling fixed points A, B of g, respectively.

By Lemma 2.1 of [14], Δ_v^* is identified with the maximal element of $\mathscr{U}_v$ containing $g^{-1}(\mathbf{H} \backslash \Delta_v)$. In cases (1) and (2) above, we denote $\{X_v, Y_v\} = \partial\Delta_v \cap \mathbf{S}^1$.

If (1) occurs, Δ_v covers both attracting and repelling fixed points of g. Moreover, the boundary $\partial\Delta_v$ is disjoint from $\text{axis}(g)$. Such an Ω_v is referred to as a type (I) region. We say that Ω_v is supported on $\mathcal{L}$ (resp. $\mathcal{R}$) if $(\mathbf{H} \backslash \Delta_v) \cap \mathbf{S}^1 \subset \mathcal{L}$ (resp. $(\mathbf{H} \backslash \Delta_v) \cap \mathbf{S}^1 \subset \mathcal{R}$). It is clear that for a type (I) region Ω_v, either $\{X_v, Y_v\} \in \mathcal{L}$ or $\{X_v, Y_v\} \in \mathcal{R}$, depending on whether Ω_v is supported on $\mathcal{L}$ or on $\mathcal{R}$.

Let $\{\Delta_k'\}$ be the half planes as defined in (3.2). It is known that Δ_k' are maximal elements of $\mathscr{U}_k'$. In what follows, we write "$X_v, Y_v < Q_k$" to mean that $\Omega_v \subset \Delta_k'$ is of type (I) and is supported on $\mathcal{L}$. Likewise, "$X_v, Y_v < P_k$" means that $\Omega_v \subset \Delta_k'$ is of type (I) and is supported on $\mathcal{R}$.

In Figure 4(a), a type (I) region Ω_v and the maximal element $\Delta_v \in \mathscr{U}_v$ are drawn, where Ω_v is supported on $\mathcal{L}$ and is contained in Δ'_k. In this case, we have $X_v, Y_v < Q_k$.



(a) (b)

Figure 4

If (2) occurs, then axis(g) crosses $\partial\Delta_v$ and $\partial\Delta^*_v$ and $\Omega_v \subset \mathbf{H} \backslash (\Delta_v \cup \Delta^*_v)$. Such an Ω_v is referred to as a type (II) region. See Figure 4(b). In this case, X_v and Y_v are separated by axis(g). As always, we let $X_v \in \mathcal{L}$ and $Y_v \in \mathcal{R}$. By our convention, $\Omega_v \subset \Delta'_k$ if and only if $X_v < Q_k$ and $Y_v < P_k$.

It is known that Ω'_0 and Ω'_m are type (II) regions. Also, in both cases (1) and (2), we remark that Δ_v covers the attracting fixed point *A* of *g*, while for every $k \geq 1$, Δ'_k covers the point *B*, but does not cover the point *A*.

The result can be summarized as the following lemma for future reference:

Lemma 4.1. *There exists a maximal element* $\Delta_v \in \mathscr{U}_v$ *with these properties*:

(i) Δ_v *covers the attracting fixed point A of g,*

(ii) Δ_v *is not contained in* Δ'_k *for any* $k \geq 0$, *and*

(iii) Δ_v *is not disjoint from* $\operatorname{axis}(g)$.

Recall that for every non-negative integer k, $u_k \in \mathcal{C}_0(S)$ is associated with the configuration $(\tau'_k, \Omega'_k, \mathscr{U}'_k) = (g^k \tau'_0 g^{-k}, g^k(\Omega'_0), g^k(\mathscr{U}'_0))$ and $\Delta'_k \in \mathscr{U}_k$ is a maximal element such that $\{P_k, Q_k\} = \partial\Delta'_k \cap \mathbf{S}^1$.

Lemma 4.2. *Let* Ω_v *be a type* (II) *region. Let* Δ_v *be as in Lemma* 4.1. *Suppose* $d_{\mathcal{C}}(v, u_k) = 1$. *Then for some* $k \geq 1$, $Q_k \leq X_v$ *and* $P_k \leq Y_v$ (*which is equivalent to that* $\Delta_v \subseteq \mathbf{H} \backslash \Delta'_k$).

Proof. Assume that $X_v < Q_k$. By Lemma 4.1, Δ_v is not contained in Δ'_k and $\Delta_v \cap \Delta'_k \neq \varnothing$. If $\partial\Delta_v \cap \partial\Delta'_k \neq \varnothing$, i.e., $P_k < Y_v$, then since $\varrho : \mathbf{H} \to \tilde{S}$ is a local homeomorphism, $\tilde{v}$ intersects $\tilde{u}_k$. Hence v intersects u_k as well, and this contradicts the hypothesis.

If $Y_v < P_k$, then $\partial\Delta_v \cap \partial\Delta'_k = \varnothing$ and $\Delta_v \cup \Delta'_k = \mathbf{H}$. See Figure 1(b). Note that Δ_v and Δ^*_v cover the attracting and repelling fixed points of g, respectively, and $\Delta_v \cap \Delta^*_k \neq \varnothing$. From the construction, $\Omega_v \subset \mathbf{H} \backslash (\Delta_v \cup \Delta^*_v)$. Hence Ω_v is disjoint from $\mathbf{H} \backslash \Delta'_k$. But $\Omega'_k \subset \mathbf{H} \backslash \Delta'_k$. We conclude that Ω_v is disjoint from (but not adjacent to) Ω'_k. If $\tilde{v} = \tilde{u}_k$, that is, $\Omega_v, \Omega'_k \in \mathscr{R}_{\tilde{v}}$, then from Lemma 2.2, $d_{\mathcal{C}}(v, u_k) \geq 2$. If $\tilde{v} \neq \tilde{u}_k$, then from Lemma 2.4, we also conclude that $d_{\mathcal{C}}(v, u_k) \geq 2$. The case where $Y_v = P_k$ cannot occur. Hence $Q_k \leq X_v$. The same argument also yields that $P_k \leq Y_v$. □

Lemma 4.3. *Let* Ω_v *be a type* (I) *region. Assume that* $d_{\mathcal{C}}(v, u_k) = 1$. *Then* $\Delta'_k \subset \Delta_v$.

Proof. We only handle the case where Ω_v is supported on $\mathcal{L}$ and show that $Q_k < X_v$ and $Q_k < Y_v$. If $X_v = Q_k$, then $Y_v \neq Q_k$. Let $g_k \in G$ be a primitive hyperbolic element such that $\operatorname{axis}(g_k)$ is the geodesic joining P_k and Q_k. Notice that $(\tau_v, \Omega_v, \mathscr{U}_v)$ is the configuration corresponding to v. It is apparent that $\partial\Delta_v$ is the axis of another primitive hyperbolic element h of G, where $h \neq g_k$. Since $Q_k = X_v$, h and g_k share the same fixed point Q_k, which contradicts that G is a discrete group. We conclude that $X_v \neq Q_k$.

Suppose now that $X_v < Q_k$. If $Y_v = Q_k$, the same argument as above leads to a contradiction. If $Y_v < Q_k$, then $\overline{\Omega}_v \cap \overline{\Omega}_k = \emptyset$, contradicting that $d_{\mathcal{C}}(v, u_k) = 1$. If $Y_v > Q_k$, then $\partial\Delta_s$ intersects $\partial\Delta'_k$. This implies that $\tilde{v}$ intersects $\tilde{u}_k = \tilde{u}_0$. Again, this contradicts that $d_{\mathcal{C}}(v, u_k) = 1$. This proves $Q_k < X_v$. Similarly, one can prove $Q_k < Y_v$. □

We proceed to study a geodesic path (1.2) in $\mathcal{C}_1(S)$ that connects u_0 and u_m for any $m \geq 1$. From the definition, we know that $d_{\mathcal{C}}(u_0, v_1) = 1$, $d_{\mathcal{C}}(v_s, u_m) = 1$ and $d_{\mathcal{C}}(v_j, v_{j+1}) = 1$ for $j = 1, \ldots, s-1$. Since u_0, u_m, and all v_j are non-preperipheral, these vertices are associated with configurations. As we have seen, $(\tau'_0, \Omega'_0, \mathscr{U}'_0)$ and $(\tau'_m, \Omega'_m, \mathscr{U}'_m)$ are the configurations for u_0 and u_m, respectively. Let $(\tau_j, \Omega_j, \mathscr{U}_j)$ be the configurations corresponding to those v_j for $j = 1, \ldots, s$.

The following lemma is a direct consequence of Lemma 4.3 and Lemma 4.2 (by setting $v = v_s$ and $k = m$):

Lemma 4.4. *With the same notation and terminology as above, we have*

(i) *if* Ω_s *is a type* (II) *region, then* $P_m \leq Y_s$ *and* $Q_m \leq X_s$, *and*

(ii) *if* Ω_s *is a type* (I) *region and is supported on* $\mathcal{L}$, *then* $Q_m < X_s$ *and* $Q_m < Y_s$. □

A question arises as to whether $\Delta_{s_0} \subseteq \mathbf{H} \backslash \Delta'_k$ implies $s = s_0$. The answer to the question is negative. However, for a type (II) region Ω_{s_0}, we have the following result.

Lemma 4.5. *Let* Ω_{s_0} *be the first type* (II) *region in the list* $\{\Omega_0 = \Omega'_0, \Omega_1, ..., \Omega_s, \Omega'_m\}$ *satisfying* $P_m \leq Y_{s_0}$.

(i) *If* $P_m = Y_{s_0}$, *then* $X_{s_0} = Q_m$. *In this case,* $s = s_0$.

(ii) *If* $P_m < Y_{s_0}$, *then* $X_{s_0} \neq Q_m$. *In this case,* $s \geq s_0 + 1$ *if* $X_{s_0} < Q_m$, *and* $s \geq s_0 \geq m+1$ *if* $X_{s_0} > Q_m$ *and all regions prior to* Ω_{s_0} *are type* (II) *regions*.

Proof. (i) If $Y_{s_0} = P_m$, by the same argument of Lemma 4.3, we conclude that $X_{s_0} = Q_m$. Thus, $\partial\Delta'_m$ is a component of $\partial\Omega_{s_0}$, which means that all boundary components of Ω_{s_0} project to the same $\varrho(\partial\Delta'_m) = \tilde{u}_0$. This in turn implies that $\Omega_{s_0}, \Omega'_m \in \mathscr{R}_{\tilde{u}_0}$. Note that Ω_{s_0} and Ω'_m share the common boundary component $\partial\Delta'_m$. It follows that Ω_{s_0} is adjacent to Ω'_m. Hence from Lemma 2.2, $d_{\mathcal{C}}(v_{s_0}, u_m) = 1$. In particular, v_{s_0} and u_m are disjoint, which says that $s = s_0$.

(ii) Assume that $P_m < Y_{s_0}$. Again by the same argument of Lemma 4.3, we see that $X_{s_0} \neq Q_m$. If $X_{s_0} < Q_m$, then by Lemma 4.2, $d_{\mathcal{C}}(v_{s_0}, u_m) \geq 2$. Hence $s \geq s_0 + 1$.

We now turn to the case where $Y_{s_0} > P_m$, $X_{s_0} > Q_m$, and all regions prior to Ω_{s_0} are type (II) regions. We claim that $X_{s_0 - 1} > Q_{m-1}$ and $Y_{s_0 - 1} > P_{m-1}$. Indeed, the assumption says that Ω_{s_0} is a type (II) region. So there exist maximal elements $\Delta_{s_0}, \Delta^*_{s_0} \in \mathscr{U}_{s_0}$ so that $\Omega_{s_0} \subset \mathbf{H} \backslash (\Delta_{s_0} \cup \Delta^*_{s_0})$, where $\Delta^*_{s_0}$ contains the geodesic joining Q_{m-1} and P_{m-1}. In other words, we have $\Omega_{s_0} \cap \Delta'_{m-1} = \varnothing$.

On the other hand, the pair $(\Delta_{s_0-1}, \Delta^*_{s_0-1})$ of maximal elements of $\mathscr{U}_{s_0-1}$ is chosen so that $\Omega_{s_0-1} \subset \mathbf{H} \backslash (\Delta_{s_0-1} \cup \Delta^*_{s_0-1})$. Since $\tilde{v}_{s_0}$ is disjoint from $\tilde{v}_{s_0-1}$, $\partial\Omega_{s_0-1}$ is disjoint from $\partial\Omega_{s_0}$. So if $X_{s_0-1} < Q_{m-1}$, we must have $\Omega_{s_0-1} \cap \Omega_{s_0} = \varnothing$. This contradicts Lemma 2.4, proving that $X_{s_0-1} > Q_{m-1}$. Similarly, one can show that $Y_{s_0-1} > P_{m-1}$. By an induction argument, one similarly shows that $X_{s_0-j} > Q_{m-j}$ and $Y_{s_0-j} > P_{m-j}$ for every j with $0 \leq j \leq m$. We conclude that $s \geq s_0 \geq m+1$. □

5. Consecutive Vertices in a Geodesic Path in the Curve Complex

In this section, we investigate consecutive vertices in a geodesic $[u_0, v_1, ..., v_s, u_m]$ in $\mathcal{C}_1(S)$ connecting u_0 and u_m. Consider again the sequence

$$\Omega_0 = \Omega'_0, \Omega_1, ..., \Omega_s, \Omega'_m. \tag{5.1}$$

Notice that Ω_0 and Ω'_m are type (II) regions, and any other region in (5.1) is either a type (I) or a type (II) region. Unless otherwise stated, in what follows, we assume that the first type (I) region in (5.1) is supported on $\mathcal{L}$.

Let Ω_{j-1}, Ω_j be two consecutive regions in (5.1), and let $\Delta_j \in \mathscr{U}_j$ and $\Delta_{j-1} \in \mathscr{U}_{j-1}$ be the maximal elements obtained from Lemma 4.1. The geodesics $\partial\Delta_j$ and $\partial\Delta_{j-1}$ intersect $\mathbf{S}^1$ at $\{X_j, Y_j\}$ and $\{X_{j-1}, Y_{j-1}\}$, respectively. Recall that $X_j, X_{j-1} \in \mathcal{L}$ and $Y_j, Y_{j-1} \in \mathcal{R}$.

Lemma 5.1. *Assume that Ω_j is a type* (I) *region and Ω_{j-1} is a type* (II) *region with $X_{j-1} < Q_k$ and $Y_{j-1} < P_k$. Then $X_j, Y_j < P_k$ if Ω_j is supported on $\mathcal{R}$; and $X_j, Y_j < Q_k$ if Ω_j is supported on $\mathcal{L}$.*

Proof. The condition $d_{\mathcal{C}}(v_{j-1}, v_j) = 1$ implies that $d_{\mathcal{C}}(\tilde{v}_{j-1}, \tilde{v}_j) = 1$. If $\tilde{v}_{j-1} \neq \tilde{v}_j$, by Lemma 2.4, $\Omega_j \bigcap \Omega_{j-1} \neq \emptyset$. Since $\Omega_{j-1} \subset \mathbf{H}\backslash(\Delta_{j-1} \bigcup \Delta^*_{j-1})$ and since $\partial\Omega_j$ and $\partial\Omega_{j-1}$ are mutually disjoint, we deduce that $\Omega_j \subset \mathbf{H}\backslash(\Delta_{j-1} \bigcup \Delta^*_{j-1})$. The assumption also tells us that $\Omega_{j-1} \subset \Delta'_k$, which means that $\Delta_{j-1} \bigcup \Delta'_k = \mathbf{H}$ and $\partial\Delta_{j-1} \bigcap \partial\Delta'_k = \emptyset$. Therefore, $\Omega_j \subset \Delta'_k$, which says $X_j, Y_j < P_k$ if Ω_j is supported on $\mathcal{R}$, or X_j, $Y_j < Q_k$ if Ω_j is supported on $\mathcal{L}$.

If $\tilde{v}_{j-1} = \tilde{v}_j = \tilde{v}$, then $v_j, v_{j-1} \in F_{\tilde{v}}$ and thus $\Omega_j, \Omega_{j-1} \in \mathscr{R}_{\tilde{v}}$. By Lemma 2.2, Ω_j and Ω_{j-1} are adjacent. It is obvious that $\overline{\Omega}_j \bigcap \overline{\Omega}_{j-1} \neq \partial\Delta_{j-1}$; otherwise, Ω_j is of type (II), which contradicts the hypothesis. It follows that $\Omega_j \subset \Delta'_k$, as asserted. □

Lemma 5.2. *Suppose that Ω_{j-1} and Ω_j are both type* (II) *regions such that $X_{j-1} < Q_k$ and $Y_{j-1} < P_{k_0}$ for some positive integers k and k_0. Then $X_j < Q_{k+1}$ and $Y_j < P_{k_0+1}$.*

Proof. The proof is essentially the same as in Lemma 4.5(ii), and the details are omitted. □

Lemma 5.3. *Suppose that Ω_{j-1} is a type* (I) *region but Ω_j is a type* (II) *region. Then $X_{j-1}, Y_{j-1} < Q_k$ implies that $X_j < Q_{k+1}$. Similarly, X_{j-1}, $Y_{j-1} < P_k$ implies that $Y_j < P_{k+1}$.*

Proof. Suppose first that $\tilde{v}_j \neq \tilde{v}_{j-1}$ are disjoint. Notice that $\overline{\Omega}_j$ is in fact the complement of all maximal elements of $\mathscr{U}_j$. Since Ω_j is of type (II), $\Omega_j \subset \mathbf{H}\backslash(\Delta_j \bigcup \Delta^*_j)$, where, as usual, $\Delta^*_j \in \mathscr{U}_j$ denotes the maximal element that contains $g^{-1}(\mathbf{H}\backslash\Delta_j)$.

Suppose that $\mathbf{H}\backslash\Delta_j$ is not included in Δ'_{k+1}. The condition that v_{j-1} is disjoint from v_j implies that all boundary components of Ω_j are disjoint from all boundary components of Ω_{j-1}. Note that $\Omega_j \subset \mathbf{H}\backslash(\Delta_j \cup \Delta^*_j)$ and $\Omega_{j-1} \subset \mathbf{H}\backslash(\Delta_{j-1} \cup \Delta^*_{j-1})$. By Lemma 3.1 and Lemma 2.1 of [14], Ω_{j-1} must be disjoint from Ω_j. But this contradicts Lemma 2.4. It follows that $\mathbf{H}\backslash\Delta_j \subset \Delta'_{k+1}$. Therefore, $\Omega_j \subset \Delta'_{k+1}$.

Similarly, we can handle the case in which $\tilde{v}_j = \tilde{v}_{j-1}$. □

For each type (II) region Ω_j, let α_j denote the angle between $\operatorname{axis}(g)$ and $\partial\Delta_j$.

Lemma 5.4. *With the same condition of Lemma* 5.3. *Assume that* X_{j-1}, $Y_{j-1} < Q_k$ *and that* $\alpha_j \geq \delta_q$ *for some integer q. Then* $X_j < Q_{k+1}$ *and* $Y_j < P_{k+1+q}$.

Proof. By applying Lemma 5.3, we conclude that $X_j < Q_{k+1}$. Suppose that $P_{k+1+q} < Y_j < P_{k+2+q}$. Observe that δ_q is the angle between $\operatorname{axis}(g)$ and the geodesic joining from Q_0 and P_q, which is also the angle between $\operatorname{axis}(g)$ and the geodesic joining from Q_{k+1} to P_{k+1+q}. We deduce that $Q_{k+j} < X_j < Q_{k+1+j}$ for some $j \geq 1$. This leads to a contradiction. It follows that $Y_j \leq P_{k+1+q}$. But certainly, $Y_j \neq P_{k+1+q}$. We conclude that $Y_j < P_{k+1+q}$, and hence $\Omega_j \subset \Delta'_{k+1+q}$, as asserted. □

We now consider the case where there are consecutive type (I) regions in the list (5.1).

Lemma 5.5. *Suppose that* Ω_{j-1} *and* Ω_j *are both type* (I) *regions. Then* $X_{j-1}, Y_{j-1} < Q_k$ *implies* $X_j, Y_j < Q_{k+1}$. *Similarly,* $X_{j-1}, Y_{j-1} < P_k$ *implies* $X_j, Y_j < P_{k+1}$.

Proof. Again, we only treat the case where $\tilde{v}_{j-1} \neq \tilde{v}_j$ are disjoint. From Lemma 2.4, we know that $\Omega_j \cap \Omega_{j-1} \neq \emptyset$ and all the boundary components of Ω_{j-1} are disjoint from all the boundary components of Ω_j. This implies that $\partial\Delta_j$ is disjoint from $\partial\Delta_{j-1}$. Hence either $\Delta_{j-1} \subset \Delta_j$ or $\Delta_j \subset \Delta_{j-1}$. In the former case, we have $\Omega_j \subset \Delta'_k$.

In the later case, we consider the half plane $\mathbf{H}\backslash\Delta_j$ and notice that $\Omega_j \subset \mathbf{H}\backslash\Delta_j$ and that $\mathbf{H}\backslash\Delta_j$ contains $\mathbf{H}\backslash\Delta_{j-1}$. Since Ω_j is a type (I) region, $\mathbf{H}\backslash\Delta_j$ is disjoint from $\operatorname{axis}(g)$. So if $X_j > Q_{k+1}$ and $X_{j-1}, Y_{j-1} < P_k$, then $\mathbf{H}\backslash\Delta_j$ would cover both Q_k and Q_{k+1}. But this contradicts Lemma 3.2. □

Finally, if there are r, $2 \leq r \leq j$, consecutive type (I) regions in the list (5.1), Lemma 5.5 can be extended to the following result.

Lemma 5.6. *Suppose that* $\Omega_{j-r+1}, \ldots, \Omega_j$ *are* r *consecutive type* (I) *regions in the sequence* (5.1) *such that* $X_{j-r+1}, Y_{j-r+1} < Q_k$. *Then* $X_j, Y_j < Q_{k+d}$, *where* $d = [r/2]$ *is the largest integer less than or equal to* $r/2$.

Proof. The assumption tells us that Ω_{j-r+1} is supported on $\mathcal{L}$. We claim that Ω_{j-r+2} is also supported on $\mathcal{L}$. Otherwise, $\overline{\Omega}_{j-r+1}$ is disjoint from $\overline{\Omega}_{j-r+2}$. By Lemma 2.4, we deduce that $d_{\mathcal{C}}(v_{j-r+1}, v_{j-r+2}) \geq 2$. This leads to a contradiction. An induction argument yields that all $\Omega_{j-r+1}, \ldots, \Omega_j$, must also be supported on $\mathcal{L}$.

The fact that $X_{j-r+2}, Y_{j-r+2} < Q_{k+1}$ follows from Lemma 5.5. Suppose $r \geq 3$ and consider the type (I) region Ω_{j-r+3} and the associated maximal element $\Delta_{j-r+3} \in \mathscr{U}_{j-r+3}$. It is clear that either $\Delta_{j-r+2} \subset \Delta_{j-r+3}$ or $\Delta_{j-r+3} \subset \Delta_{j-r+2}$. In either cases, since $\tilde{v}_{j-r+3} \in \mathcal{C}_0(\tilde{S})$ is simple, by Lemma 3.2, $(\mathbf{H}\backslash\Delta_{j-r+3}) \cap \mathbf{S}^1$ cannot cover $(Q_k Q_{k+1})$, which implies

$X_{j-r+3}, Y_{j-r+3} < Q_{k+1}$. But then we must have $X_{j-r+4}, Y_{j-r+4} < Q_{k+2}$ (if $r \geq 4$). Here we notice that $\tilde{v}_{j-r+4} \in \mathcal{C}_0(\tilde{S})$ is a simple geodesic, and by Lemma 3.2 again, $(\mathbf{H} \backslash \Delta_{j-r+4}) \cap \mathbf{S}^1$ cannot cover $(Q_{k+1}Q_{k+2})$.

It follows from an induction argument that $X_{j-r+q}, Y_{j-r+q} < Q_{k+[q/2]}$ for $1 \leq q \leq r$. Setting $q = r$ we conclude that $X_j, Y_j < Q_{k+d}$ for $d = [r/2]$.

□

6. First Three Vertices in a Geodesic Path Joining u_0 and u_m

Observe that $\Omega_0 = \Omega_0'$ is a type (II) region that is contained in $\mathbf{H} \backslash (\Delta_0 \cup \Delta_0')$. Since $\tilde{u}_0$ is chosen so that $i(\tilde{u}_0, \tilde{c}) \geq 2$, $\Delta_0 \cup \Delta_1' = \mathbf{H}$ and $\partial\Delta_0 \cap \partial\Delta_1' = \emptyset$. It follows that $Y_0 < P_1$ and $X_0 < Q_1$.

Suppose that Ω_1 is a type (I) region supported on $\mathcal{L}$. By Lemma 5.1, $X_1, Y_1 < Q_1$. This implies $\Omega_1 \subset \Delta_1'$.

If Ω_1 is a type (II) region, then since Ω_0 is of type (II), by Lemma 5.2, $Y_1 < P_2$ and $X_1 < Q_2$. This means $\Omega_1 \subset \Delta_2'$.

By combining these two possibilities for Ω_1, we conclude that $\Omega_1 \subset \Delta_2'$.

Now take Ω_2 into consideration. If both Ω_1 and Ω_2 are type (I) regions, then $X_1, Y_1 < Q_1$, and by Lemma 5.5, $X_2, Y_2 < Q_2$, which says $\Omega_2 \subset \Delta_2'$.

If both Ω_1 and Ω_2 are type (II) regions, then by Lemma 5.2, $Y_2 < P_3$ and $X_2 < Q_3$, which says $\Omega_2 \subset \Delta_3'$.

If Ω_1 is a type (II) region but Ω_2 is a type (I) region, then by Lemma 5.1, $X_2, Y_2 < P_2$ or $X_2, Y_2 < Q_2$, both of which imply that $\Omega_2 \subset \Delta_2'$.

In the case where Ω_1 is of type (I) and Ω_2 is of type (II), we have $X_1, Y_1 < Q_1$. By Lemma 5.3, $X_2 < Q_2$. We need to rule out the possibility that the other end Y_2 of $\partial\Delta_2$ is pretty far down, such as $Y_2 \in \mathcal{R}$ is near to the point A. Notice that $(\mathbf{H}\backslash\Delta_1) \cap \mathbf{S}^1 \subset (Q_0 Q_1)$.

Since S is a surface with type $(p, 1)$, we have $\mathcal{C}_0(S) = \hat{\mathcal{C}}_0(S)$, which tells us that every vertex in $\mathcal{C}_0(S)$ is non-preperipheral. This in turn implies that $\tilde{v}_1 \subset \tilde{S}$ is a simple closed geodesic. Recall that $(\tau_1, \Omega_1, \mathscr{U}_1)$ is the configuration corresponding to v_1, and that $\tilde{c} = \varrho(\text{axis}(g))$ is a filling closed geodesic that intersects $\tilde{v}_1$. Hence $\text{axis}(g)$ intersects some geodesics in $\{\varrho^{-1}(\tilde{v}_1)\}$. Let $\gamma_1 \in \{\varrho^{-1}(\tilde{v}_1)\}$ be such a geodesic. Since $\varrho(\partial\Delta_0') = \varrho(\partial\Delta_1')$ $= \tilde{u}_0$ is disjoint from $\varrho(\gamma_1) = \tilde{v}_1$ and since $\{\varrho^{-1}(\tilde{v}_1)\} \subset \mathbf{H}$ consists of mutually disjoint geodesics, we conclude that $g^i(\gamma_1)$ are disjoint from $\partial\Delta_0'$ and $\partial\Delta_1'$.

Observe also that all geodesics $g^i(\gamma_1)$ are disjoint and intersect $\text{axis}(g)$. These geodesics are also disjoint from $\partial\Delta_2$. Let β_1 denote the angle between $\text{axis}(g)$ and γ_1 (which is also the angle between $\text{axis}(g)$ and any $g^i(\gamma_1)$), and let $\delta(\gamma_1)$ denote the associated angle as defined in Section 3. Then $[\delta(\gamma_1)] = \delta_2$, where we define $[\alpha] = \delta_{j+1}$ if $\delta_{j+1} \leq \alpha < \delta_j$. By Lemma 3.3, $\beta_1 \geq \delta_1$. Thus by Lemma 3.3 again, $\alpha_2 \geq \delta(\gamma_1) \geq \delta_2$, where we recall α_2 is the angle between $\text{axis}(g)$ and $\partial\Delta_2$. Since $X_2 < Q_2$, by Lemma 5.4, we obtain $Y_2 < P_4$. That is, $\Omega_2 \subset \Delta_4'$.

By combining all the possibilities for Ω_1 and Ω_2, we conclude that $\Omega_2 \subset \Delta_4'$. From Lemma 4.2 we thus obtain the following result.

Proposition 6.1. *Let* $u_0 \in \mathcal{C}_0(S)$ *and let* $(\tau_0', \Omega_0', \mathscr{U}_0')$ *be the corresponding configuration. Let* $\tilde{c} \subset \tilde{S}$ *be a filling closed geodesic determined by a pseudo-Anosov map* $f = g^* \in \mathscr{F}$ *for an essential hyperbolic element* $g \in G$, *with the properties that* $i(\tilde{c}, \tilde{u}_0) \geq 2$ *and* $\Omega_0' \cap \text{axis}(g) \neq \emptyset$. *Then for any* $m \geq 2$, *any geodesic path* $[u_0, v_1, \ldots, v_s, f^m(u_0)]$

in $\mathcal{C}(S)$ *joining* u_0 *and* $f^m(u_0)$, *we have* $\Omega_1 \subset \Delta_2'$ *and* $\Omega_2 \subset \Delta_4'$, *where* Ω_1, Ω_2 *correspond to* v_1 *and* v_2, *respectively. Consequently, it holds that* $d_{\mathcal{C}}(u_0, f^2(u_0)) \geq 3$ *and* $d_{\mathcal{C}}(u_0, f^4(u_0)) \geq 4$.

Remark 4. In fact, for any Riemann surface S of type (p, n) with $3p + n > 4$, the results in [14] and [16] show that for $m = 3$ or 4, $d_{\mathcal{C}}(u_0, f^m(u_0)) \geq m$ for any $u_0 \in \mathcal{C}_0(S)$ and any pseudo-Anosov map $f \in \mathscr{F}$.

7. Proof of Theorem 1.2

Following the notation and terminology introduced in Section 3, we know that $\tilde{c} = \varrho(\text{axis}(g))$ is an oriented filling closed geodesic on $\tilde{S}$. Thus $i(\tilde{c}, \tilde{u}) \geq 1$ for any $\tilde{u} \in \mathcal{C}_0(\tilde{S})$. Choose $\tilde{u}_0$ so that $i(\tilde{u}_0, \tilde{c}) \geq 2$.

Refer to Figure 2. A geometric observation reveals that $\partial\Delta_0 = \partial\Delta_1'$ if and only if $i(\tilde{u}_0, \tilde{c}) = 1$. Thus the condition $i(\tilde{u}_0, \tilde{c}) \geq 2$ guarantees that $\partial\Delta_0$ lies in between $\partial\Delta_0'$ and $\partial\Delta_1'$. Recall that the closure of Ω_0' is the complement of all maximal elements of $\mathscr{U}_0'$. We have $\Omega_0' \subset \mathbf{H} \setminus (\Delta_0 \cup \Delta_0')$ and hence also $\Omega_m' \subset \mathbf{H} \setminus \Delta_m'$.

We only prove the result for $m > 0$. It is trivial that $d_{\mathcal{C}}(u_0, f(u_0)) \geq 1$. By Proposition 6.1, $d_{\mathcal{C}}(u_0, f^m(u_0)) \geq m$ for $m = 2, 4$. By Theorem 1.1 of [14], $d_{\mathcal{C}}(u_0, f^3(u_0)) \geq 3$. So we assume that $m \geq 5$. As in Section 3, we assume that (1.2) is a geodesic path in $\mathcal{C}_1(S)$ that connects u_0 and u_m. Again, let $(\tau_j, \Omega_j, \mathscr{U}_j)$ be the configurations corresponding to v_j for $j = 1, ..., s$.

Let Δ_s be the component of $\mathbf{H} \setminus \overline{\Omega}_s$ obtained from Lemma 4.1. By Lemma 4.4, we know that $\Delta_s \subset \mathbf{H} \setminus \Delta_m'$, which is equivalent to that $\{X_s, Y_s\} = \mathbf{S}^1 \cap \partial\Delta_s$ lies outside of $\Delta_m' \cap \mathbf{S}^1$.

First we suppose that $\Omega_1, ..., \Omega_s$ in (5.1) are type (I) regions. For any two successive regions Ω_j, Ω_{j+1}, where $1 \leq j \leq s-1$, we denote the components of $\mathbf{H} \backslash \Omega_j$ and $\mathbf{H} \backslash \Omega_{j+1}$ by Δ_j and Δ_{j+1}, respectively, which are obtained from Lemma 4.1. Then Lemma 2.4 asserts that either $\Delta_j \subset \Delta_{j+1}$ or $\Delta_{j+1} \subset \Delta_j$. As a consequence, if Ω_j is supported on $\mathcal{L}$, then so is Ω_{j+1}. We see that all Ω_j are supported on $\mathcal{L}$. It is also readily seen that $\Delta_s \cap \Delta'_m \neq \emptyset$. By Lemma 4.2, $d_{\mathcal{C}}(v_s, u_m) \geq 2$ unless $X_s > Q_m$. Suppose that $X_s > Q_m$. Since $X_1, Y_1 < Q_1$, from Lemma 5.6, $X_s < Q_{[s/2]+1}$. It follows that $Q_m < X_s < Q_{[s/2]+1}$. Hence $m + 1 \leq [s/2] + 1 \leq s/2 + 1$, which gives $s \geq 2m$.

We assume throughout the section that there is at least one type (II) region among $\Omega_1, ..., \Omega_s$. We rewrite the sequence (5.1) as

$$\Omega_{p(0)} = \Omega'_0, \Gamma_{p(0)}, \Omega_{p(1)}, \Gamma_{p(1)}, ..., \Omega_{p(M)}, \Gamma_{p(M)}, \Omega'_m, \tag{7.1}$$

where $M \geq 1$ and $\Omega_{p(i)}$, $0 \leq i \leq M$, are all type (II) regions and $\Gamma_{p(i)}$ consists of consecutive type (I) regions.

Note that some $\Gamma_{p(i)}$ could be empty. However, if $\Gamma_{p(i)} \neq \emptyset$, we can write $\Gamma_{p(i)} = \{\omega_{p(i)+1}, ..., \omega_{p(i)+r(i)}\}$, where each $\omega_{p(i)+j}$ is a type (I) region and is contained in $\mathbf{H} \backslash \Delta_{p(i)+j}$. Here we recall that $\Delta_{p(i)+j}$ is the component of $\mathbf{H} \backslash \overline{\omega}_{p(i)+j}$ containing the fixed points of g. Hence $\omega_{p(i)+1}$ is disjoint from $\mathrm{axis}(g)$. By the same argument as above, for any two successive regions $\omega_{p(i)+j}, \omega_{p(i)+j+1} \in \Gamma_{p(i)}$, they both are supported on $\mathcal{L}$ or on $\mathcal{R}$. Since elements in $\Gamma_{p(i)}$ are connected by a path, we see that all elements in $\Gamma_{p(i)}$ are supported on $\mathcal{L}$ or on $\mathcal{R}$.

By assumption, every $\omega \in \Gamma_{p(0)}$ is supported on $\mathcal{L}$. Let $p(i+1)$

$(\leq p(M))$ be the largest integer such that either $r(j) = 0$ for $j \leq i$ or every ω_r, $r < p(i+1)$, is supported on $\mathcal{L}$. Consider now the sub-collection $\{\Omega_{p(i)}, \Gamma_{p(i)}, \Omega_{p(i+1)}\}$ in (7.1). If $\Gamma_{p(i)} = \emptyset$; that is, $r(i) = 0$, then by Lemma 5.2, $Y_{p(i+1)} < Y_{p(i)+1}$ and $X_{p(i+1)} < X_{p(i)+1}$.

Suppose that $r(i) > 0$. It is known that $\partial\Delta_{p(i)+1}$ projects (under the universal covering map $\varrho : \mathbf{H} \to \tilde{S}$) to a simple closed geodesic $\tilde{v}_{p(i)+1}$ on $\tilde{S}$. Since $\tilde{c} = \varrho(\text{axis}(g))$ is filling closed geodesic, $\tilde{v}_{p(i)+1}$ must intersect $\tilde{c}$, which means that there is a geodesic $\gamma_{p(i)+1}$ in $\{\varrho^{-1}(\tilde{v}_{p(i)+1})\}$ that intersects $\text{axis}(g)$. Let $\beta_{p(i)+1}$ denote the angle between $\text{axis}(g)$ and $\gamma_{p(i)+1}$. Let $\alpha_{p(i)}$ denote the angle between $\text{axis}(g)$ and $\partial\Delta_{p(i)}$. Since $\partial\Delta_{p(i)}$ is disjoint from $\{g^i(\gamma_{p(i)+1}) : i \geq 0\}$, by Lemma 3.4, $\beta_{p(i)+1} \geq \delta_{a(i)+1}$, where $a(i)$ is the number that satisfies

$$a(0) = 1, \text{ and } \delta_{a(i)} \leq \alpha_{p(i)} \leq \delta_{a(i)-1} \text{ for all } i \geq 0.$$

For each $j = 1, \ldots, p(i) + r(i) - 1$, there is $g_j \in G$ in the conjugacy class of g such that $\text{axis}(g_j)$ intersects both $\partial\Delta_{p(i)+j}$ and $\partial\Delta_{p(i)+j+1}$. Let $h_j \in G$ be such that $h_j(\text{axis}(g_j)) = \text{axis}(g)$. Observe that all the angle values are invariant under h_j-translations. We see that there is a geodesic $\gamma_{p(i)+2}$ in $\{\varrho^{-1}(\tilde{v}_{p(i)+2})\}$ that intersects $\text{axis}(g)$. Let $\beta_{p(i)+2}$ be the angle between $\text{axis}(g)$ and $\gamma_{p(i)+2}$, which is also the angle between $\text{axis}(g_1)$ and $h_1^{-1}(\gamma_{p(i)+2})$.

Evidently, $\gamma_{p(i)+2}$ is disjoint from $\{g^i(\gamma_{p(i)+1}) : i \geq 0\}$. Since $\text{axis}(g)$ is an invariant geodesic under the action of g, $\beta_{p(i)+1}$ is also the angle between $\text{axis}(g)$ and any $g^i(\gamma_{p(i)+1})$. Since $\beta_{p(i)+1} \geq \delta_{a(i)+1}$, by applying

Lemma 3.4 again, we see that $\beta_{p(i)+2} \geq \delta_{a(i)+2}$, and so on, this process can continue through all elements in $\Gamma_{p(i)}$, and we conclude that $\beta_{p(i)+r(i)} \geq \delta_{a(i)+r(i)}$. By applying Lemma 3.4 once again for the geodesics $\gamma_{p(i)+r(i)}$ and $\partial\Delta_{p(i+1)}$, we obtain

$$\alpha_{p(i+1)} \geq \delta_{a(i)+r(i)+1}, \tag{7.2}$$

where, as usual, $\alpha_{p(i+1)}$ denotes the angle between $\operatorname{axis}(g)$ and $\partial\Delta_{p(i+1)}$. If $r(i) = 0$, (7.2) becomes $\alpha_{p(i+1)} \geq \delta_{a(i)+1}$. Hence from the definition of $a(i)$, we get $\delta_{a(i+1)} \geq \delta_{a(i)+r(i)+1}$, which means that $a(i+1) \leq a(i) + r(i) + 1$. Thus, an easy computation yields the following inequality:

$$a(i) \leq \sum_{j=0}^{i-1}(r(j)+1) + a(0) = \sum_{j=0}^{i-1}(r(j)+1) + 1. \tag{7.3}$$

Recall that our assumption guarantees that all members in $\Gamma_{p(0)}$ (if not empty) are supported on $\mathcal{L}$, and that $p(i+1)$ is the integer such that either $r(j) = 0$ for $j \leq i$ or every ω_r, $r < p(i+1)$, are supported on $\mathcal{L}$. We claim that

$$X_{p(i+1)} < Q_{\sigma(i)+1}, \text{ and} \tag{7.4}$$

$$Y_{p(i+1)} < P_{\sigma(i)+1+a(i)+r(i)+1}, \tag{7.5}$$

where $\sigma(i) = \sum_{j=0}^{i}[r(j)/2]$.

We prove (7.4) by induction. First, Lemma 5.6 and Lemma 5.2 assert that $X_{p(1)} < Q_{\sigma(0)+1}$, where $\sigma(0) = [r(0)/2]$. Suppose that $X_{p(i)} < Q_{\sigma(i-1)+1}$. If $r(i) \neq 0$, then since $v_{p(i)}$ is disjoint from $v_{p(i)+1}$, and $v_{p(i)+1}$ corresponds to $\omega_{p(i)+1}$ which is of type (I), from Lemma 2.4 and Lemma 3.1, we know that $X_{p(i)+1}, Y_{p(i)+1} < Q_{\sigma(i-1)+1}$. By Lemma 5.5, $X_{p(i)+2}, Y_{p(i)+2} <$

$Q_{\sigma(i-1)+1}$, and so on, by the same argument of Lemma 5.6, we deduce that $X_{p(i)+r(i)}, Y_{p(i)+r(i)} < Q_{\sigma(i-1)+[r(i)/2]}$. It follows from Lemma 2.4 and Lemma 3.1 that $X_{p(i+1)} < Q_{\sigma(i-1)+[r(i)/2]+1}$. But it is easy to verify that $\sigma(i-1) + [r(i)/2] = \sigma(i)$. If $r(i) = 0$, then clearly, $\sigma(i-1) = \sigma(i)$ and thus $X_{p(i+1)} < Q_{\sigma(i-1)+1} = Q_{\sigma(i)+1}$. Hence (7.4) is established. (7.5) follows from (7.2), (7.4) and Lemma 5.4.

Rewrite (7.5) as

$$Y_{p(i+1)} < P_{\lambda(i)}, \text{ where } \lambda(i) = \sum_{j=0}^{i}\left(\left[\frac{r(j)}{2}\right] + r(j) + 1\right) + 2. \tag{7.6}$$

By the definition of $p(i+1)$, we know that $\Gamma_{p(i+1)} \neq \varnothing$ (i.e. $r(j+1) > 0$), and all regions in $\Gamma_{p(i+1)}$ are supported on $\mathcal{R}$. By calculations similar to the above, we obtain

$$X_{p(i+2)} < Q_{\nu(i+1)} \text{ and } Y_{p(i+2)} < P_{\mu(i+1)},$$

where

$$\mu(i+1) = \lambda(i) + \left[\frac{r(i+1)}{2}\right]$$

and

$$\nu(i+1) = \sigma(i) + \left[\frac{r(i+1)}{2}\right] + (r(i+1) + 1).$$

By comparing the functions λ, σ, μ and ν, we find that

$$\lambda(i+1) - 2 \geq \max\{\sigma(i+1), \mu(i+1), \nu(i+1)\}.$$

In general, for $q \geq i+1$, let $\eta(q) = \sum_{j=0}^{q} b(j) + 2$, where $b(j)$ is either $[r(j)/2] + r(j) + 1$ or $[r(j)/2]$ depending on whether $\Gamma_{p(j)}$ is supported on

$\mathcal{L}$ or on $\mathcal{R}$. If $\lambda(q) \neq \eta(q)$, there is at least one j_0 such that $\Gamma_{p(j_0)}$ is supported on $\mathcal{R}$, i.e., $b(j_0) = [r(j_0)/2]$ and $r(j_0) \geq 1$. This means that $\lambda(q) - \eta(q) \geq r(j_0) + 1 \geq 2$. Thus we also have

$$\lambda(q) - 2 \geq \eta(q). \tag{7.7}$$

Henceforth, by virtue of Lemma 4.4, (7.4), (7.5) and (7.7), one may assume, without loss of generality, that $P_m \leq Y_{p(i)}$ or $Q_m \leq X_{p(i)}$ for some integer i (the situation where $Y_{p(M)} < P_m$ and $X_{p(M)} < Q_m$ is more optimal. See Addendum). We also see that, in order to achieve the goal of minimizing the number $p(i)$ for which $P_m \leq Y_{p(i)}$ or $Q_m \leq X_{p(i)}$, it is enough to only estimate the smallest integer L for which $P_m \leq Y_{p(L)}$ under the assumption that all ω_j in (7.1) are supported on $\mathcal{L}$.

Consider here a special case where all $r(i) = 0$ for $0 \leq i \leq L - 1$; i.e., (5.1) consists of type (II) regions only prior to the region $\Omega_{p(L)}$. Since $\{X_0, Y_0\} = \partial\Delta_0 \cap \mathbf{S}^1$ and $i(\tilde{u}_0, \tilde{c}) \geq 2$, by the same argument of Lemma 4.5 (ii), we see that $Y_{m-1} < P_m$ and $X_{m-1} < Q_m$. In particular, by Lemma 4.2, $d_{\mathcal{C}}(v_{m-1}, u_m) \geq 2$, which implies that $s \geq m$. Therefore, $d_{\mathcal{C}}(u_0, f^m(u_0)) \geq m + 1$. So this scenario is not optimal.

It remains to consider the case where some $r(j) \neq 0$. We claim that $P_m < Y_{p(L)}$. Suppose $P_m = Y_{p(L)}$. By the same argument of Lemma 4.5, $Q_m = X_{p(L)}$ and thus $\Omega_{p(L)}$ is adjacent to Ω'_m. On the other hand, since $r(j) \neq 0$ for some $j < L$, from the calculation above, we deduce that $X_{p(L)} < Q_m$. This leads to a contradiction. Hence $P_m \neq Y_{p(L)}$ and thus $P_m < Y_{p(L)}$.

Along with (7.6), we obtain

$$P_m < Y_{p(L)} < P_{\lambda(L-1)}.$$

There are two cases.

Case 1. $X_{p(L)} < Q_m$. Let s_0 be specified as in Lemma 4.5, that is, $s_0 = \sum_{j=0}^{L-1} r(j) + L$. By Lemma 4.5, we have $s \geq s_0 + 1 = \sum_{j=0}^{L-1} r(j) + L + 1$, which tells us that

$$s - L - 1 \geq \sum_{j=0}^{L-1} r(j). \tag{7.8}$$

Let $K > 0$ be the number of zeros in $\{r(0), r(1), \ldots, r(L-1)\}$. Then $K \leq L - 1$ and there are $L - K$ nonzero integers in $\{r(0), r(1), \ldots, r(L-1)\}$, which yields the following:

$$\sum_{j=0}^{L-1} r(j) = \sum_{j=0}^{L-1} \{r(j) : r(j) \neq 0\} \geq L - K. \tag{7.9}$$

Clearly, $P_m < P_{\lambda(L-1)}$ implies $m < \lambda(L-1)$. It then follows from (7.6) that

$$\begin{aligned} m + 1 &\leq \lambda(L-1) \\ &= K + \sum_{j=0}^{L-1} \left\{ \left[\frac{r(j)}{2} \right] + (r(j) + 1) : r(j) \neq 0 \right\} + 2 \\ &\leq K + 2 + \frac{3}{2} \left(\sum_{j=0}^{L-1} \{r(j) : r(j) \neq 0\} \right) + (L - K). \end{aligned} \tag{7.10}$$

Hence (7.10) and (7.8) combine to yield

$$m + 1 \leq K + 2 + \frac{3(s - L - 1)}{2} + L - K = \frac{3s}{2} - \frac{L}{2} + \frac{1}{2}. \tag{7.11}$$

Since $L \geq 1$, we obtain

$$s \geq \frac{2(m+1)}{3}. \tag{7.12}$$

Case 2. $X_{p(L)} > Q_m$. If all $r(j) = 0$, by Lemma 4.5(ii), $d_{\mathcal{C}}(u_0, u_m) > m$. Hence one may assume that $r(j) \neq 0$ for some j. In this case, $s \geq p(L) = \sum_{j=0}^{L-1} r(j) + L$. Thus

$$L - K \leq \sum_{j=0}^{L-1} r(j) \leq s - L. \tag{7.13}$$

In particular, (7.13) implies that

$$1 \leq L \leq \frac{s + K}{2}. \tag{7.14}$$

By assumption, $r(j_0) \neq 0$ for some j_0, which says $r(j_0) + 1 \geq 2$. From the argument similar to (7.7), (7.10) and (7.11) we conclude that

$$m + 1 \leq \lambda(L - 1) - 2 \leq K + 1 + \frac{3}{2}\left(\sum_{j=0}^{L-1} \{r(j) : r(j) \neq 0\}\right) + L - K. \tag{7.15}$$

It follows from (7.13), (7.14) and (7.15) that

$$s \geq \frac{2m + 3}{3}. \tag{7.16}$$

Combining (7.12) and (7.16), we deduce that $d_{\mathcal{C}}(u_0, f^m(u_0)) = s + 1 \geq \frac{2m + 5}{3}$ whenever $m \geq 5$.

In particular, when $m = 5$, we have $d_{\mathcal{C}}(u_0, f^5(u_0)) \geq \frac{15}{3} = 5$. When $m = 6$, we have $d_{\mathcal{C}}(u_0, f^6(u_0)) \geq \frac{17}{3} = 5.666\cdots$. So $d_{\mathcal{C}}(u_0, f^6(u_0)) \geq 6$. When $m = 7$, we have $d_{\mathcal{C}}(u_0, f^7(u_0)) \geq \frac{19}{3} = 6.333\cdots$. So $d_{\mathcal{C}}(u_0, f^7(u_0)) \geq 7$.

Addendum

Here we consider the case in which $Y_{p(M)} < P_m$ and $X_{p(M)} < Q_m$. Then $s = \sum_{j=0}^{M} r(j) + M$, where $M \geq 1$ and $r(M) > 2$. This simplifies to

$$s - M = \sum_{j=0}^{M} r(j). \tag{0.1}$$

On the other hand, by a similar discussion in Case 1, we can obtain

$$m + 1 \leq \lambda(M - 1) + \left[\frac{r(M)}{2}\right].$$

This gives

$$\begin{aligned} m + 1 &\leq \sum_{j=0}^{M-1}\left(\left[\frac{r(j)}{2}\right] + r(j) + 1\right) + 2 + \left[\frac{r(M)}{2}\right] \\ &\leq \frac{3}{2}\left(\sum_{j=0}^{M-1} r(j)\right) + M + 2 + \frac{r(M)}{2} \\ &= \frac{3}{2}\left(\sum_{j=0}^{M} r(j)\right) + M + 2 - r(M). \end{aligned} \tag{0.2}$$

Since $r(M) \geq 3$, (0.2) and (0.1) combine to yield

$$m + 1 \leq \frac{3(s - M)}{2} + M + 2 - 3 = \frac{3s}{2} - \frac{M}{2} - 1. \tag{0.3}$$

Recall that $M \geq 1$. It follows from (0.3) that

$$\frac{3s}{2} \geq m + 2 + \frac{M}{2} \geq m + \frac{5}{2}.$$

Hence

$$s \geq \frac{2m + 5}{3}.$$

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