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NEW SOLUTIONS FOR BIOLOGICAL MODEL

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Abstract

This work considers the Liouville equation, the sinh-Gordon equation and special form of Zhiber-Shabat equation, arising in mathematical biology. The extended sine-cosine method has been used in order to obtain multiple exact special solutions. The proposed scheme can be applied to a wider class of nonlinear equations.

1. Introduction

Many phenomena in physics and other fields such as biology, chemistry, and mechanics are described by nonlinear partial differential equations (NLPDEs).

As mathematical models of the phenomena, the investigation of exact solutions of NLPDEs will help one to understand these phenomena better.

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Also, the explicit formulas may provide information and help us to understand the mechanism of related models.

In recent years, many powerful and efficient methods to find analytic solutions of nonlinear equations have been presented by a diverse group of scientists. These methods include the standard tanh and extended tanh methods [14-16, 20], the standard and extended sine-cosine methods [18, 19, 21], the $\sec^P$-$\tanh^p$ method [3], the standard exp-function method and extended F-expansion method [5], the $\exp(-f(x))$-expansion method [10], the Adomian decomposition method [9], the Jacobi elliptic function expansion method [8], Hirota's bilinear transformation [22], the (G'/G)-expansion method, the modified simple equation method [6, 7], the reproducing kernel method [2], and the functional variable method [4]. Practically, there is no unified method that can be used to handle all types of nonlinear problems which arise in phenomena.

This work is related with the class of nonlinear partial equations

$$u_{xt} + f(u) = 0, \tag{1}$$

which plays a significant role in mathematical biology and many scientific applications such as solid state physics, nonlinear optics, plasma physics, fluid dynamics, dislocations in crystals, kink dynamics, and chemical kinetics, quantum field theory, the propagation of fluxions in Josephson junctions [1, 5-7, 10-17]. The function $f(u)$ takes many forms such as

$$f(u) = \begin{cases} pe^u + qe^{-u}, \\ pe^u, \\ qe^{-u}. \end{cases}$$

The first and second one characterize the sinh-Gordon equation when $p = 1, q = -1,$ and the Liouville equation when $p = 1,$ respectively. The third characterizes a special case of the Zhiber-Shabat equation.

The nonlinear Zhiber-Shabat equation takes the form

$$u_{xt} + pe^{u} + qe^{-u} + re^{-2u} = 0, \tag{2}$$

where p, q and r are arbitrary constants.

In this paper, the extended sine-cosine method is used to solve equation (1) and also equation (2). The mentioned method is based on the explicit linearization of NLEEs for travelling waves with a certain substitution which leads to a second-order differential equation with constant coefficients. Moreover, the algorithm that was used here is also a computerized method, in which generated algebraic system is solved with computer packages.

The aim of this paper is to derive more exact solitons, travelling wave and exponential solutions of the sinh-Gordon equation, the Liouville equation, and their general forms, and Zhiber-Shabat equation when $p = 0$ and $r = 0$.

The extended sine-cosine method presented in [18, 19, 21] will be employed to solve these equations.

Some entirely new exact solutions of the Liouville, sinh-Gordon and the Zhiber-Shabat equations are obtained.

The rest of the paper is organized as follows: in Section 2, we describe briefly the extended sine-cosine method and the way it is used to derive the solutions of nonlinear PDEs. In Section 3, the method mentioned in Section 2 is used to find the exact solutions of the sinh-Gordon equation and its general form. In Section 4, the method has been applied to the Liouville equation and its general form. In Section 5, the method is used to find the exact solutions of the Zhiber-Shabat equation when $p = 0$ and $r = 0$. Conclusion is provided in the last Section 6.

2. Outline of Extended sine-cosine Method

This section aims to outline the use of the extended sine-cosine method to solve the nonlinear partial differential equations (PDEs) based on the method given in [18, 19, 21]. Given nonlinear PDE in two independent variables

$$P(u, u_t, u_x, u_{xx}, u_{xt}, \ldots) = 0, \tag{3}$$

where P is in general nonlinear function of its variables, the subscripts denote the partial derivatives. The main steps of this method are as follows:

Step 1. Upon using new variables $\xi = kx + wt$, $u(x, t) = u(\xi)$, where k and w are arbitrary constants, equation (3) can be carried into ordinary differential equations (ODEs) in one independent variable

$$Q(u, u', u'', \ldots) = 0, \tag{4}$$

where Q is a polynomial in $u(\xi)$ and its total derivative, while prime denotes derivative with respect to ξ. Equation (4) is then integrated as long as all derivatives, where the associated integration constants can be determined.

Step 2. Suppose the travelling wave solution of equation (4) can be expressed as follows:

$$u(x, t) = U(\zeta) = \sum_{i=0}^{M} a_i (v(\xi))^i, \tag{5}$$

where a_i, $i = 0, 1, 2, \ldots, M$ are constants to be determined and $v = v(\xi)$ satisfies a nonlinear ordinary differential equation of the first order

$$v' = \frac{dv}{d\xi} = \varepsilon\sqrt{a + bv^2}, \; a, b \in R, \; \varepsilon = \pm 1. \tag{6}$$

Equation (6) has the following general solutions of seven kinds:

$$v(\zeta) = \varepsilon\sqrt{\frac{a}{b}} \sinh\sqrt{b}(\xi + \xi_0), \; a > 0, \; b > 0, \; \varepsilon = \pm 1, \tag{7a}$$

$$v(\zeta) = \sqrt{\frac{a}{-b}} \sin\sqrt{-b}(\xi + \xi_0), \; a > 0, \; b < 0, \; \varepsilon = 1, \tag{7b}$$

$$v(\zeta) = \sqrt{\frac{a}{-b}} \cos \sqrt{-b}(\xi + \xi_0), \; a > 0, \; b < 0, \; \varepsilon = -1, \tag{7c}$$

$$v(\zeta) = \sqrt{\frac{-a}{b}} \cosh \sqrt{b}(\xi + \xi_0), \; a < 0, \; b > 0, \; \varepsilon = \pm 1, \tag{7d}$$

$$v(\zeta) = e^{\varepsilon \sqrt{b}(\xi + \xi_0)}, \; a = 0, \; b > 0, \; \varepsilon = \pm 1, \tag{7e}$$

$$v(\zeta) = c_1 \sin \sqrt{-b}(\xi + \xi_0) + c_2 \cos \sqrt{-b}(\xi + \xi_0), \; a = 0, \; b < 0, \; \varepsilon = \pm 1, \tag{7f}$$

$$v(\zeta) = \varepsilon \sqrt{a}(\xi + \xi_0), \; a > 0, \; b = 0, \; \varepsilon = \pm 1, \tag{7g}$$

where ξ_0 is an arbitrary constant and the multiple special solutions of nonlinear PDE (3) are obtained by using (5) and (7).

Step 3. The balance constant M can be selected by the analysis of the leading term.

Step 4. Substituting (5) into (4), using (6), and collecting all the terms of the same power v^i, $i = 0, 1, 2, 3, \ldots$ and equating them to zero, a system of nonlinear algebraic equations on a_i, $i = 0, 1, 2, \ldots, M$, a, b, k and w is obtained which can be solved by Maple or Mathematica to get all the constants a_i, $i = 0, 1, 2, \ldots, M$, a, b, k and w.

Step 5. Substituting these values and the solutions of equation (7) into equation (5), the exact solutions of equation (3) is obtained.

3. The General sinh-Gordon Equation

First, we consider the general form of the sinh-Gordon equation. Let $r = 0$ in equation (2). Then

$$u_{xt} + pe^{u} + qe^{-u} = 0. \tag{8}$$

Apply the extended sine-cosine method.

Step 1. Let us consider the travelling wave solutions $u(x, t) = U(\xi)$, $\xi = kx + wt$. Then equation (8) becomes

$$kwu'' + pe^{u} + qe^{-u} = 0. \tag{9}$$

Applying the Painlevé property:

$$v = e^{u}, \text{ or } u = \ln(v), \tag{10}$$

and we get $u' = \frac{1}{v}v'$, and $u'' = \frac{1}{v}v'' - \frac{1}{v^2}(v')^2$. Equation (9) turns to

$$wk(vv'' - (v')^2) + pv^3 + qv = 0. \tag{11}$$

Step 2. Suppose that the solution of equation (11) can be expressed by a polynomial in terms of v as in (5).

Step 3. By the analysis of the leading term, we select the balance constant $M = 1$. Thus, the extended sine-cosine method gives the finite expansion

$$u(x, t) = U(\xi) = a_0 + a_1 v. \tag{12}$$

Step 4. Substituting (12) into (11) and using equation (6), we get

$$a_0^3 p + a_0 q + 3a_0^2 a_1 pv + a_1 qv + 3a_0 a_1^2 pv^2 + a_1^3 pv^3 - aa_0 kw\zeta^2 - aa_1 kwv\zeta^2$$

$$+ a_0 a_1 bkwv\zeta^2 - a_0 bkwv^2\zeta^2 + a_1^2 bkwv^2\zeta^2 - a_1 bkwv^3\zeta^2 = 0,$$

and with $\zeta^2 = 1$, equating each coefficient of this polynomial to zero yields a set of simultaneous algebraic equations for a_0, a_1, k, w, p and q as follows:

$$(v)^0 : a_0^3 p + a_0 q - kwaa_0 = 0,$$

$$(v)^1 : 3a_0^2 a_1 p + a_1 q - kwaa_1 + kwba_0 a_1 = 0,$$

$$(v)^2 : 3a_0 a_1^2 p - kwba_0 + kwba_1^2 = 0,$$

$$(v)^3 : a_1^3 p - kwba_1 = 0. \tag{13}$$

Solving system (13), with the aid of Mathematica, we get the following:

$$a_0 = -2\frac{akw - q}{bkw},\ a_1 = \pm 2\sqrt{\frac{akw - q}{bkw}} \text{ and } p = \frac{(kwb)^2}{4(akw - q)}, \tag{14}$$

where $\frac{akw - q}{bkw} > 0$ and $bkw \neq 0.$

Step 5. By combining equations (7a)-(7g), (14) and (12), we obtain the following:

Case 1.

(i) $a > 0,\ b > 0.$ This gives the soliton solutions

$$v(x, t) = -2\frac{akw - q}{bkw} \pm 2\sqrt{\frac{akw - q}{bkw}}\sqrt{\frac{a}{b}} \sinh\sqrt{b}(kx + wt + \xi_0),$$

where k, w are two arbitrary real parameters such that $akw - q > 0$ and $kw > 0,$ or $akw - q < 0$ and $kw < 0.$

In view of these results, and noting that $u(x, t) = \ln v(x, t),$ we obtain the soliton solutions:

$$u(x, t) = \ln\left(-2\frac{akw - q}{bkw} \pm 2\sqrt{\frac{akw - q}{bkw}}\sqrt{\frac{a}{b}} \sinh\sqrt{b}(kx + wt + \xi_0)\right),$$

where k, w are two arbitrary real parameters such that $akw > q$ and $kw > 0,$ or $akw < q$ and $kw < 0,$ and ξ_0 is an arbitrary constant.

(ii) $a < 0,\ b > 0.$ This gives the soliton solutions

$$u(x, t) = \ln\left(-2\frac{akw - q}{bkw} \pm 2\sqrt{\frac{akw - q}{bkw}}\sqrt{\frac{-a}{b}} \cosh\sqrt{b}(kx + wt + \xi_0)\right),$$

where $akw > q$ and $kw > 0,$ or $akw < q$ and $kw < 0,$ and ξ_0 is an arbitrary constant.

Case 2.

$a > 0,\ b < 0.$ This gives the travelling wave solutions

$$u(x, t) = \ln\left(-2\frac{akw - q}{bkw} \pm 2\sqrt{\frac{akw - q}{bkw}}\sqrt{\frac{a}{-b}}\sin\sqrt{-b}(kx + wt + \xi_0)\right),$$

where $akw > q$ and $kw < 0$, or $akw < q$ and $kw > 0$, and ξ_0 is an arbitrary constant, and

$$u(x, t) = \ln\left(-2\frac{akw - q}{bkw} \pm 2\sqrt{\frac{akw - q}{bkw}}\sqrt{\frac{a}{-b}}\cos\sqrt{-b}(kx + wt + \xi_0)\right),$$

where $akw > q$ and $kw < 0$, or $akw < q$ and $kw > 0$, and ξ_0 is an arbitrary constant.

Case 3.

$a = 0,\ b > 0.$ This gives the exponential solutions

$$u(x, t) = \ln\left(2\frac{q}{bkw} \pm 2\sqrt{\frac{-q}{bkw}}\, e^{\pm\sqrt{b}(kx+wt+\xi_0)}\right),$$

where $q > 0$ and $kw < 0$, or $q < 0$ and $kw > 0$, and ξ_0 is an arbitrary constant.

Case 4.

$a = 0,\ b < 0.$ This gives the travelling wave solutions

$$u(x, t)$$

$$= \ln\left(\frac{2q}{bkw} \pm 2\sqrt{\frac{-q}{bkw}}(c_1 \sin\sqrt{-b}(kx + wt + \xi_0) + c_2 \cos\sqrt{-b}(kx + wt + \xi_0))\right),$$

where $q > 0$ and $kw > 0$, or $q < 0$ and $kw < 0$, c_1 and c_2 are arbitrary real constants, and ξ_0 is an arbitrary constant.

Now, consider the sinh-Gordon equation

$$u_{xt} + e^{u} - e^{-u} = 0. \tag{15}$$

As stated before, by using Step 1 of extended sine-cosine method and the Painlevé property (10), equation (14) is reduced to the following ordinary differential equations:

$$wk(vv'' - (v')^2) + v^3 + v = 0. \tag{16}$$

Now, we use Step 2 of extended sine-cosine method.

Suppose that the solution of equation (16) can be expressed by a polynomial in terms of v as in (5). Through Step 3 to Step 5, we get, by the analysis of the leading term, the balance constant $M = 1$. Thus, the extended sine-cosine method gives the finite expansion

$$u(x, t) = U(\xi) = a_0 + a_1 v. \tag{17}$$

Substituting (17) into (16), using equation (6), and equating each coefficient of this polynomial to zero yields a set of simultaneous algebraic equations for determining a_0, a_1, a, b, k and w as in nonlinear system (13) with $p = 1$, $q = -1$. Solving the new system with the aid of Mathematica, the following was found:

$$b = \frac{a_1^2}{kw}, \; a = \frac{-4 + a_1^4}{4kw} \text{ and } a_0 = -\frac{a_1^2}{2}, \tag{18}$$

where $a_1 \neq 0$, and $kw \neq 0$.

By combining equations (7a), (7g), (18) and (17), we obtain the following:

Case 1.

(i) $a > 0, b > 0$. This gives the soliton solutions

$$u(x, t) = \ln\left(-\frac{a_1^2}{2} \pm a_1 \sqrt{\frac{-4 + a_1^4}{4a_1^2}} \sinh \sqrt{\frac{a_1^2}{kw}}(kx + wt + \xi_0)\right),$$

where $kw > 0$ and $-\sqrt{2} > a_1$, or $a_1 > \sqrt{2}$, and ξ_0 is an arbitrary constant.

(ii) $a < 0,\ b > 0.$ This gives the soliton solutions

$$u(x, t) = \ln\left(-\frac{a_1^2}{2} + a_1\sqrt{-\frac{-4 + a_1^4}{4a_1^2}} \cosh\sqrt{\frac{a_1^2}{kw}}(kx + wt + \xi_0)\right),$$

where $kw > 0$ and $-\sqrt{2} < a_1 < \sqrt{2},$ and ξ_0 is an arbitrary constant.

Case 2.

$a > 0,\ b < 0.$ This gives the travelling wave solutions

$$u(x, t) = \ln\left(-\frac{a_1^2}{2} + a_1\sqrt{-\frac{-4 + a_1^4}{4a_1^2}} \sin\sqrt{-\frac{a_1^2}{kw}}(kx + wt + \xi_0)\right),$$

where $kw < 0$ and $-\sqrt{2} < a_1 < \sqrt{2},$ and ξ_0 is an arbitrary constant, and

$$u(x, t) = \ln\left(-\frac{a_1^2}{2} + a_1\sqrt{-\frac{-4 + a_1^4}{4a_1^2}} \cos\sqrt{-\frac{a_1^2}{kw}}(kx + wt + \xi_0)\right),$$

where $kw < 0$ and $-\sqrt{2} < a_1 < \sqrt{2},$ and ξ_0 is an arbitrary constant.

Case 3.

(i) $a = 0,\ b > 0.$ This gives the exponential solutions

$$u(x, t) = \ln(-1 \pm \sqrt{2}e^{\pm\sqrt{\frac{2}{kw}}(kx+wt+\xi_0)}),$$

where $kw > 0,$ and ξ_0 is an arbitrary constant.

(ii) $b < 0.$ This gives the travelling wave solutions

$$u(x, t)$$

$$= \ln\left(-1 \pm \sqrt{2}\left(c_1 \sin\sqrt{-\frac{2}{kw}}(kx + wt + \xi_0) + c_2 \cos\sqrt{-\frac{2}{kw}}(kx + wt + \xi_0)\right)\right),$$

$kw < 0,$ c_1 and c_2 are arbitrary real constants, and ξ_0 is an arbitrary constant.

4. The General Form of Liouville Equation

For the nonlinear Zhiber-Shabat equation (2), if p is an arbitrary constant, $q = 0$ and $r = 0$, then

$$u_{xt} + pe^{u} = 0. \tag{19}$$

As stated before, by using Step 1 of extended sine-cosine method and the Painlevé property (10), equation (19) is reduced to the following ordinary differential equation:

$$wku'' + pe^{u} = 0. \tag{20}$$

Applying the Painlevé property (10), equation (20) turns to

$$wk\left(vv'' - (v')^{2}\right) + pv^{3} = 0. \tag{21}$$

Now, use Step 2 of extended sine-cosine method. Suppose that the solution of equation (19) can be expressed by a polynomial in terms of v as in (5). Then through Step 3 to Step 5, we get, by the analysis of the leading term, the balance constant $M = 1$. Thus, the extended sine-cosine method gives the finite expansion

$$u(x, t) = U(\xi) = a_0 + a_1 v. \tag{22}$$

Substituting (22) into (21) and using equation (6), and equating each coefficient of this polynomial to zero yields a set of simultaneous algebraic equations for determining a_1, a_0, k, w, a, b and p as follows:

$$(v)^{0} : a_0^{3} p - kwaa_0 = 0,$$

$$(v)^{1} : 3a_0^{2} a_1 p - kwaa_1 + kwba_0 a_1 = 0,$$

$$(v)^{2} : 3a_0 a_1^{2} p - kwba_0 + kwba_1^{2} = 0,$$

$$(v)^{3} : a_1^{3} p - kwba_1 = 0. \tag{23}$$

Solving the system (23), we get the following:

$$b = \frac{a_1^2 p}{kw}, \; a = \frac{a_1^4 p}{4kw} \text{ and } a_0 = -\frac{a_1^2}{2}, \tag{24}$$

where $kw \neq 0$, p and a_1 are left as free parameters.

In view of these results, note that $u(x, t) = \ln v(x, t)$.

However, for $a_0 > 0$, we obtain the solutions as follows:

Case 1.

$a > 0$, $b > 0$. This gives the soliton solutions

$$u(x, t) = \ln\left(-\frac{a_1^2}{2} \pm a_1\sqrt{\frac{a_1^2}{4}} \sinh\sqrt{\frac{a_1^2 p}{kw}}(kx + wt + \xi_0)\right),$$

where $kw > 0$ and $p > 0$, or $kw < 0$ and $p < 0$, and ξ_0 is an arbitrary constant.

Case 2.

$a > 0$, $b < 0$. This gives the travelling wave solutions

$$u(x, t) = \ln\left(-\frac{a_1^2}{2} + a_1\sqrt{\frac{a_1^2}{4}} \sin\sqrt{-\frac{a_1^2 p}{kw}}(kx + wt + \xi_0)\right),$$

where $kw < 0$ and $p > 0$, or $kw > 0$ and $p < 0$, and ξ_0 is an arbitrary constant.

5. Solution of Equation $u_{xt} + qe^{-u} = 0$

For the nonlinear Zhiber-Shabat equation (2), if q is an arbitrary constant, $p = 0$ and $r = 0$, then

$$u_{xt} + qe^{-u} = 0. \tag{25}$$

As stated before, by using Step 1 of the extended sine-cosine method and Painlevé property (10), equation (25) is reduced to the following ordinary differential equations:

$$wk(vv'' - (v')^2) + qv = 0. \tag{26}$$

Next, use Step 2 of the extended sine-cosine method. Suppose that the solution of equation (26) can be expressed by a polynomial in terms of v as in (5).

By the analysis of the leading term, the balance constant $M = 2$. Thus, the extended sine-cosine method gives the finite expansion

$$u(x, t) = U(\xi) = a_0 + a_1 v + a_2 v^2. \tag{27}$$

Substituting (27) into (26), using equation (6), and equating each coefficient of this polynomial to zero yields a set of simultaneous algebraic equations for determining a_0, a_1, a_2, a, b, k, w and q as follows:

$(v)^0 : a_0 q - kwaa_0 + 2aa_0 a_2 kw = 0,$

$(v)^1 : a_1 q - kwaa_1 + 2kwaa_1 a_2 + kwba_0 a_1 = 0,$

$(v)^2 : a_2 q - kwaa_2 + 2kwaa_2^2 - kwa_0 b + kwba_1^2 + 4a_0 a_2 bkw = 0,$

$(v)^3 : -kwa_1 b + 5kwa_1 a_2 b = 0,$

$$(v)^4 : -kwa_2 b + 4kwba_2^2 = 0. \tag{28}$$

Solving equation (25), with the aid of Mathematica, we get the following:

First: for $a_2 = \dfrac{1}{4}$,

$a_1 = 0$, and $a = \dfrac{2q}{kw}$, a_0, b, $k \neq 0$, $w \neq 0$ and q are arbitrary constants, and we obtain the following as solutions.

Case 1.

(i) $a > 0,\ b > 0.$ This gives the soliton solutions

$$u(x,\ t) = \ln\left(a_0 + \frac{1}{4}\left(\sqrt{\frac{2q}{bkw}}\sinh\sqrt{b}(kx + wt + \xi_0)\right)^2\right),$$

where $kw > 0$ and $q > 0,$ or $kw < 0$ and $q < 0,$ a_0 and ξ_0 are arbitrary constants.

(ii) $a < 0,\ b > 0.$ This gives the soliton solutions

$$u(x,\ t) = \ln\left(a_0 + \frac{1}{4}\left(\sqrt{\frac{-2q}{bkw}}\cosh\sqrt{b}(kx + wt + \xi_0)\right)^2\right),$$

where $kw > 0$ and $q < 0,$ or $kw < 0$ and $q > 0,$ a_0 and ξ_0 are arbitrary constants.

Case 2.

$a > 0,\ b < 0.$ This gives the travelling wave solutions

$$u(x,\ t) = \ln\left(a_0 + \frac{1}{4}\left(\sqrt{\frac{2q}{-bkw}}\sin\sqrt{-b}(kx + wt + \xi_0)\right)^2\right),$$

where $kw > 0$ and $q > 0,$ or $kw < 0$ and $q < 0,$ a_0 and ξ_0 are arbitrary constants, and

$$u(x,\ t) = \ln\left(a_0 + \frac{1}{4}\left(\sqrt{\frac{2q}{-bkw}}\cos\sqrt{-b}(kx + wt + \xi_0)\right)^2\right),$$

where $kw > 0$ and $q > 0,$ or $kw < 0$ and $q < 0,$ a_0 and ξ_0 are arbitrary constants.

Case 3.

$a > 0,\ b = 0.$ This gives the solutions as

$$u(x,\ t) = \ln\left(a_0 \pm \frac{1}{4}\left(\sqrt{\frac{2q}{kw}}(kx + wt + \xi_0)\right)^2\right),$$

where $kw > 0$ and $q > 0$, or $kw < 0$ and $q < 0$, a_0 and ξ_0 are arbitrary constants.

Second: for $a_2 = 0$,

$a = \frac{q}{kw}$, and $b = 0$, a_0, a_1, $k \neq 0$, $w \neq 0$ and q are arbitrary constants, we obtain the following solution:

$$u(x, t) = \ln\left(a_0 \pm a_1\left(\sqrt{\frac{q}{kw}}(kx + wt + \xi_0)\right)\right),$$

where $kw > 0$ and $q > 0$, or $kw < 0$ and $q < 0$, a_0 and ξ_0 are arbitrary constants.

It is interesting to point out that the extended sine-cosine method does not work in the Zhiber-Shabat equation when $r \neq 0$.

6. Conclusion

In this paper, the extended sine-cosine method was applied to obtain the new forms of solitons, travelling wave and exponential solutions for the sinh-Gordon equation, the Liouville equation and the Zhiber-Shabat equation when $p = r = 0$ arising in mathematical biology. We also obtained some different solutions at the same time. The method can be applied to many other nonlinear equations or systems. In addition, this method is also computerizable.

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