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NUMERICAL SOLUTION OF *m*-DIMENSIONAL STOCHASTIC ITÔ-VOLTERRA INTEGRAL EQUATIONS BY STOCHASTIC OPERATIONAL MATRIX BASED ON RATIONALIZED HAAR WAVELET

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Abstract

The multidimensional Itô-Volterra integral equations arise in many problems such as exponential population growth model with several independent white noise sources. In this paper, we obtain stochastic operational matrix of rationalized Haar functions on interval $[0, 1)$ to

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solve m-dimensional stochastic Itô-Volterra integral equations. By using rationalized Haar functions and their stochastic operational matrix of integration, m-dimensional stochastic Itô-Volterra integral equation can be reduced to a linear system which can be directly solved by Gaussian elimination method. This scheme is applied for some numerical examples in the population growth. The results show the efficiency and accuracy of the method.

1. Introduction

We know that stochastic Itô-Volterra integral equations arise in many problems in mechanics, finance, biology, medical, social sciences, etc. So the study of such problems is very useful in application and there is an increasing demand for studying the behavior of a number of sophisticated dynamical systems in physical, medical and social sciences, as well as in engineering and finance. These systems are often dependent on a noise source, on a Gaussian white noise, for example, governed by certain probability laws. So that modeling such phenomena naturally requires the use of various stochastic differential equations [1-7] or, in more complicated cases, stochastic Itô-Volterra and Itô-Volterra-Fredholm integral equations and stochastic integro-differential equations [7-17]. Because in many problems such equations of course cannot be solved explicitly, it is important, to find their approximate solutions by using some numerical methods [1-5, 13-15].

Many orthogonal functions or polynomials, such as block pulse functions, hybrid functions, Haar wavelet, Legendre wavelet, Coifman wavelet, Shannon wavelet, Daubechies wavelet, and Bernestein polynomials, were used to derive solutions of different integral equations [18-29]. Here we use the rationalized Haar wavelet and stochastic integration operational matrix for deriving solution of m-dimensional stochastic Itô-Volterra integral equation.

So, consider the following m-dimensional linear stochastic Itô-Volterra integral equation:

$$X(t) = f(t) + \int_0^t b(t, s)X(s)ds + \sum_{i=1}^{m} \int_0^t \sigma_i(t, s)X(s)dB_i(s), \quad t \in [0, T),$$

where $X(t)$, $f(t)$, $b(t, s)$ and $\sigma_i(t, s)$, $i = 1, 2, ..., m$, for $t, s \in [0, T)$, are the stochastic processes defined on the same probability space (Ω, F, P), and $X(t)$ is an unknown function. Also, $B(t) = (B_1(t), B_2(t), ..., B_m(t))$ is m-dimensional Brownian motion and $\int_0^t \sigma_i(t, s) X(s) dB_i(s)$, $i = 1, 2, ..., m$, are the Itô integrals.

This paper is organized as follows: in Section 2, we describe the basic properties of the rationalized Haar functions and functions approximation by rationalized Haar functions and integration operational matrix. In Section 3, we obtain the stochastic integration operational matrix. In Section 4, we solve stochastic Itô-Volterra integral equations with several independent white noise sources by using stochastic integration operational matrix. In Section 5, we examine the efficiency and accuracy of this method by giving some numerical examples in the population growth. Finally, Section 6 gives some brief conclusion.

2. Rationalized Haar Functions (RHFs)

The goal of this section is to recall notations and definition of the rationalized Haar functions and to recall some known results and formulas that are important for this paper. These have been discussed thoroughly in [21, 22].

2.1. Definition

The rationalized Haar functions (RHFs) are defined as:

$$RH(r, t) = \begin{cases} 1 & \dfrac{j-1}{2^i} \le t < \dfrac{j-\frac{1}{2}}{2^i}, \\ -1 & \dfrac{j-\frac{1}{2}}{2^i} \le t < \dfrac{j}{2^i}, \\ 0 & \text{otherwise}, \end{cases} \tag{1}$$

where

$$r = 2^i + j - 1, \quad i = 0, 1, 2, 3, \ldots, \quad j = 1, 2, 3, \ldots, 2^i.$$

$RH(0, t)$ is defined for $i = j = 0$ and is given by

$$RH(0, t) = 1, \quad 0 \le t < 1, \tag{2}$$

with orthogonality property

$$\int_0^1 RH(r, t) RH(v, t) dt = \begin{cases} 2^{-i} & \text{for } r = v, \\ 0 & \text{for } r \neq v, \end{cases} \tag{3}$$

where

$$v = 2^n + m - 1, \quad n = 0, 1, 2, 3, \quad \ldots, m = 1, 2, 3, \ldots, 2^n.$$

2.2. Functions approximation

A function $f(t)$ defined over the interval $t \in [0, 1)$ may be expanded in RHFs as

$$f(t) = \sum_{r=0}^{\infty} f_r RH(r, t), \tag{4}$$

where f_r, $r = 0, 1, 2, \ldots,$ are given by

$$f_r = 2^i \int_0^1 f(t) RH(r, t) dt, \tag{5}$$

with $r = 2^i + j - 1$ for $i = 0, 1, 2, 3, \ldots,$ $j = 1, 2, 3, \ldots, 2^i$ and $r = 0$ for $i = j = 0$.

If we let $i = 0, 1, 2, \ldots, \alpha,$ then the infinite series in equation (4) is truncated up to its first *k* terms as

$$f(t) \simeq \sum_{r=0}^{k-1} f_r RH(r, t) = F^T \Phi(t) = \Phi^T(t) F, \tag{6}$$

where $k = 2^{\alpha+1}$, $\alpha = 0, 1, 2, \ldots.$

The vectors of F and $\Phi(t)$ are defined as

$$F = (f_0, f_1, \ldots, f_{k-1})^T, \tag{7}$$

$$\Phi(t) = (\phi_0(t), \phi_1(t), \ldots, \phi_{k-1}(t))^T, \; \phi_r(t) = RH(r, t), \; r = 0, 1, 2, \ldots, k-1. \tag{8}$$

Let $k(t, s) \in L^2([0, 1) \times [0, 1))$. It can be similarly expanded with respect to RHFs such as

$$k(t, s) \simeq \sum_{r=0}^{k-1} \sum_{v=0}^{k-1} k_{rv} \phi_r(t) \phi_v(s) = \Phi^T(t) K \Phi(s), \tag{9}$$

where $K = (k_{rv})_{k \times k}$, and k_{rv} for $r = 0, 1, 2, \ldots, k-1$, $v = 0, 1, 2, \ldots, k-1$, is given by

$$k_{rv} = 2^{i+n} \int_0^1 \int_0^1 k(t, s) \phi_r(t) \phi_v(s), \quad i, n = 0, 1, 2, \ldots, \alpha.$$

The first eight RHFs can be written in matrix form as

$$\hat{\Phi}_{8\times 8} = \begin{pmatrix} \phi_0(t) \\ \phi_1(t) \\ \phi_2(t) \\ \phi_3(t) \\ \phi_4(t) \\ \phi_5(t) \\ \phi_6(t) \\ \phi_7(t) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}. \tag{10}$$

In equation (10), the row denotes the order of the Haar function. The matrix $\hat{\Phi}_{k\times k}$ can be expressed as

$$\hat{\Phi}_{k\times k} = \left(\Phi\left(\frac{1}{2k}\right), \Phi\left(\frac{3}{2k}\right), \Phi\left(\frac{5}{2k}\right), \ldots, \Phi\left(\frac{2k-1}{2k}\right)\right), \tag{11}$$

and using equation (6), we get

$$\left(f\left(\frac{1}{2k}\right), f\left(\frac{3}{2k}\right), f\left(\frac{5}{2k}\right), \ldots, f\left(\frac{2k-1}{2k}\right)\right) = F^T \hat{\Phi}_{k\times k}. \tag{12}$$

From equations (9) and (12), we have

$$K = (\hat{\Phi}_{k\times k}^{-1})^T \hat{K} \hat{\Phi}_{k\times k}^{-1}, \tag{13}$$

where

$$\hat{K} = (\hat{k}_{lp})_{k\times k}, \quad \hat{k}_{lp} = k\left(\frac{2l-1}{2k}, \frac{2p-1}{2k}\right), \quad l, \ p = 1, 2, 3, \ldots, k,$$

and so

$$\hat{\Phi}_{k\times k}^{-1} = \left(\frac{1}{k}\right)\hat{\Phi}_{k\times k}^{-1} \cdot diag(1, 1, 2, 2, \underbrace{2^2, \ldots, 2^2}_{2^2}, \underbrace{2^3, \ldots, 2^3}_{2^3}, \ldots, \underbrace{\frac{k}{2}, \ldots, \frac{k}{2}}_{\frac{k}{2}}). \tag{14}$$

2.3. The product operational matrix

The rationalized Haar product matrix is defined by [22]

$$\Psi_{k\times k}(t) = \Phi(t)\Phi^T(t). \tag{15}$$

Furthermore, by (1) and (2), we get

$$\phi_0(t)\phi_q(t) = \phi_0(t), \quad q = 0, 1, 2, \ldots, k-1,$$

and for $p < q$, we can write

$$\phi_p(t)\phi_q(t)$$

$$= \begin{cases} \phi_q(t) & \text{if} \quad \phi_q(t) \text{ occurs during the first positive half wave of } \varphi_p(t), \\ -\phi_q(t) & \text{if} \quad \phi_q(t) \text{ occurs during the second negative half wave of } \varphi_p(t), \\ 0 & \text{otherwise.} \end{cases} \tag{16}$$

Also, the square of any RHFs is a block pulse function, with magnitude of 1 during both the positive and negative half waves of RHFs. Thus, we get

$$\Psi_{8\times 8}(t)$$

$$= \begin{pmatrix} \phi_0 & \phi_1 & \phi_2 & \phi_3 & \phi_4 & \phi_5 & \phi_6 & \phi_7 \\ \phi_1 & \phi_0 & \phi_2 & -\phi_3 & \phi_4 & \phi_5 & -\phi_6 & -\phi_7 \\ \phi_2 & \phi_2 & \frac{\phi_0+\phi_1}{2} & 0 & \phi_4 & -\phi_5 & 0 & 0 \\ \phi_3 & -\phi_3 & 0 & \frac{\phi_0-\phi_1}{2} & 0 & 0 & \phi_6 & -\phi_7 \\ \phi_4 & \phi_4 & \phi_4 & 0 & \frac{\phi_0+\phi_1+2\phi_2}{4} & 0 & 0 & 0 \\ \phi_5 & \phi_5 & -\phi_5 & 0 & 0 & \frac{\phi_0+\phi_1-2\phi_2}{4} & 0 & 0 \\ \phi_6 & -\phi_6 & 0 & \phi_6 & 0 & 0 & \frac{\phi_0-\phi_1+2\phi_3}{4} & 0 \\ \phi_7 & -\phi_7 & 0 & -\phi_7 & 0 & 0 & 0 & \frac{\phi_0-\phi_1-2\phi_3}{4} \end{pmatrix}. \tag{17}$$

In general, we have

$$\Psi_{k\times k}(t) = \begin{pmatrix} \Psi_{\left(\frac{k}{2}\right)\times\left(\frac{k}{2}\right)}(t) & H_{\left(\frac{k}{2}\right)\times\left(\frac{k}{2}\right)}(t) \\ H^T_{\left(\frac{k}{2}\right)\times\left(\frac{k}{2}\right)}(t) & D_{\left(\frac{k}{2}\right)\times\left(\frac{k}{2}\right)}(t) \end{pmatrix}, \tag{18}$$

where

$$\Psi_{1\times 1}(t) = \phi_0(t),$$

$$H_{\left(\frac{k}{2}\right)\times\left(\frac{k}{2}\right)}(t) = \hat{\Phi}_{\left(\frac{k}{2}\right)\times\left(\frac{k}{2}\right)} \cdot diag(\phi_{\frac{k}{2}}(t), \phi_{\frac{k}{2}+1}(t), \ldots, \phi_{k-1}(t)),$$

$$D_{\left(\frac{k}{2}\right)\times\left(\frac{k}{2}\right)}(t) = diag[\hat{\Phi}^{-1}_{\left(\frac{k}{2}\right)\times\left(\frac{k}{2}\right)} \cdot (\phi_0(t), \phi_1(t), \ldots, \phi_{\frac{k}{2}-1}(t))^T]^T.$$

Furthermore, by multiplying the matrix $\Psi_{k\times k}(t)$ by the vector F in equation (7), we obtain

$$\Psi_{k\times k}(t)F = \hat{F}_{k\times k}\phi(t), \tag{19}$$

where

$$\hat{F}_{k\times k} = \begin{pmatrix} \hat{F}_{\left(\frac{k}{2}\right)\times\left(\frac{k}{2}\right)} & G_{\left(\frac{k}{2}\right)\times\left(\frac{k}{2}\right)} \\ \hat{G}_{\left(\frac{k}{2}\right)\times\left(\frac{k}{2}\right)} & \hat{D}_{\left(\frac{k}{2}\right)\times\left(\frac{k}{2}\right)} \end{pmatrix}, \tag{20}$$

with

$$\hat{F}_{1\times 1}(t) = f_0,$$

$$G_{\left(\frac{k}{2}\right)\times\left(\frac{k}{2}\right)} = \hat{\Phi}_{\left(\frac{k}{2}\right)\times\left(\frac{k}{2}\right)} \cdot diag(f_{\frac{k}{2}}, f_{\frac{k}{2}+1}, \ldots, f_{k-1}),$$

$$\hat{G}_{\left(\frac{k}{2}\right)\times\left(\frac{k}{2}\right)} = diag(f_{\frac{k}{2}}, f_{\frac{k}{2}+1}, \ldots, f_{k-1}) \cdot \hat{\Phi}^{-1}_{\left(\frac{k}{2}\right)\times\left(\frac{k}{2}\right)},$$

$$\hat{D}_{\left(\frac{k}{2}\right)\times\left(\frac{k}{2}\right)} = diag[(f_0, f_1, \ldots, f_{\frac{k}{2}-1}) \cdot \hat{\Phi}_{\left(\frac{k}{2}\right)\times\left(\frac{k}{2}\right)}].$$

2.4. Integration operational matrix

Consider the following approximation:

$$\int_0^t \Phi(s)ds \simeq P\Phi(t), \tag{21}$$

with operational matrix of integration

$$P_{k\times k} = \frac{1}{2k}\begin{pmatrix} 2kP_{\left(\frac{k}{2}\right)\times\left(\frac{k}{2}\right)} & -\hat{\Phi}_{\left(\frac{k}{2}\right)\times\left(\frac{k}{2}\right)} \\ \hat{\Phi}^{-1}_{\left(\frac{k}{2}\right)\times\left(\frac{k}{2}\right)} & 0 \end{pmatrix}, \tag{22}$$

where $\hat{\Phi}_{1\times 1} = 1$ and $P_{1\times 1} = \frac{1}{2}$.

So, we can write

$$\int_0^t f(s)ds \simeq \int_0^t F^T\Phi(s)ds \simeq F^T P\Phi(t). \tag{23}$$

Also, the integration of cross product of two RH function vectors is

$$\int_0^t \Phi(t)\Phi^T(t)dt = \widetilde{D}, \tag{24}$$

where $\widetilde{D}$ is diagonal matrix given by

$$\widetilde{D} = diag(1, 1, \frac{1}{2}, \frac{1}{2}, \underbrace{\frac{1}{2^2}, ..., \frac{1}{2^2}}_{2^2}, \underbrace{\frac{1}{2^3}, ..., \frac{1}{2^3}}_{2^3}, ..., \underbrace{\frac{1}{2^\alpha}, ..., \frac{1}{2^\alpha}}_{2^\alpha}).$$

3. Stochastic Integration Operational Matrix

Here we would like to compute the Itô integral for each $\phi_r(t)$, $r = 0, 1, 2, ..., k-1$. To illustrate the calculation procedures, first, let $\alpha = 0$ or $k = 2$. Using equations (1) and (2), we get

$$\int_0^t \phi_0(s)dB(s) = B(t) \simeq \begin{cases} B\left(\frac{1}{4}\right) & 0 \le t < \frac{1}{2}, \\ B\left(\frac{3}{4}\right) & \frac{1}{2} \le t < 1, \end{cases} \tag{25}$$

$$\int_0^t \phi_1(s)dB(s) = \begin{cases} B(t) & 0 \le t < \frac{1}{2} \\ 2B\left(\frac{1}{2}\right) - B(t) & \frac{1}{2} \le t < 1 \end{cases}$$

$$\simeq \begin{cases} B\left(\frac{1}{4}\right) & 0 \le t < \frac{1}{2}, \\ 2B\left(\frac{1}{2}\right) - B\left(\frac{3}{4}\right) & \frac{1}{2} \le t < 1. \end{cases} \tag{26}$$

We can rewrite (25) and (26), in terms of the RHFs $\phi_0(t)$ and $\phi_1(t)$, as follows:

$$\int_0^t \Phi(s)dB(s) \simeq S\Phi(t), \tag{27}$$

where 2×2 stochastic operational matrix of integration is given by

$$S_{2\times 2} = \frac{1}{2}\begin{pmatrix} B\left(\frac{1}{4}\right) + B\left(\frac{3}{4}\right) & B\left(\frac{1}{4}\right) - B\left(\frac{3}{4}\right) \\ B\left(\frac{1}{4}\right) - B\left(\frac{3}{4}\right) + 2B\left(\frac{1}{2}\right) & B\left(\frac{1}{4}\right) + B\left(\frac{3}{4}\right) - 2B\left(\frac{1}{2}\right) \end{pmatrix}. \tag{28}$$

For convenience, consider

$$\alpha_{pq} = \sum_{i=\frac{(q-1)k}{p}+1}^{\frac{qk}{p}} B\left(\frac{2i-1}{2k}\right), \quad p = 1, 2, 2^2, 2^3, \ldots, k,$$

$$q = 1, 2, 3, 4, \ldots, p, \tag{29}$$

and

$$\beta_{pq} = \alpha_{pq} - \alpha_{p,q+1}, \; p = 1, 2, 2^2, 2^3, \ldots, k, \; q = 1, 2, 3, 4, \ldots, p-1. \tag{30}$$

By using equations (29) and (30), $S_{2\times 2}$ is written by

$$S_{2\times 2} = \frac{1}{2}\begin{pmatrix} \alpha_{11} & \beta_{21} \\ \beta_{21} + 2B\left(\frac{1}{2}\right) & \alpha_{11} - 2B\left(\frac{1}{2}\right) \end{pmatrix}. \tag{31}$$

Now, we choose $\alpha = 1$ or $k = 4$. Using equations (1) and (2), we get the following consecutive relations:

$$\int_0^t \phi_0(s)dB(s) = B(t) \simeq \begin{cases} B\left(\frac{1}{8}\right) & 0 \le t < \frac{1}{4}, \\ B\left(\frac{3}{8}\right) & \frac{1}{4} \le t < \frac{1}{2}, \\ B\left(\frac{5}{8}\right) & \frac{1}{2} \le t < \frac{3}{4}, \\ B\left(\frac{7}{8}\right) & \frac{3}{4} \le t < 1, \end{cases} \tag{32}$$

$$\int_0^t \phi_1(s)dB(s) = \begin{cases} B(t) & 0 \le t < \frac{1}{2} \\ 2B\left(\frac{1}{2}\right) - B(t) & \frac{1}{2} \le t < 1 \end{cases}$$

$$\simeq \begin{cases} B\left(\frac{1}{8}\right) & 0 \le t < \frac{1}{4}, \\ B\left(\frac{3}{8}\right) & \frac{1}{4} \le t < \frac{1}{2}, \\ 2B\left(\frac{1}{2}\right) - B\left(\frac{5}{8}\right) & \frac{1}{2} \le t < \frac{3}{4}, \\ 2B\left(\frac{1}{2}\right) - B\left(\frac{7}{8}\right) & \frac{3}{4} \le t < 1, \end{cases} \tag{33}$$

$$\int_0^t \phi_2(s)dB(s) = \begin{cases} B(t) & 0 \le t < \frac{1}{4} \\ 2B\left(\frac{1}{2}\right) - B(t) & \frac{1}{4} \le t < \frac{1}{2} \\ 2B\left(\frac{1}{4}\right) - B\left(\frac{1}{2}\right) & \frac{1}{2} \le t < 1 \end{cases}$$

$$\simeq \begin{cases} B\left(\frac{1}{8}\right) & 0 \le t < \frac{1}{4}, \\ 2B\left(\frac{1}{4}\right) - B\left(\frac{3}{8}\right) & \frac{1}{4} \le t < \frac{1}{2}, \\ 2B\left(\frac{1}{4}\right) - B\left(\frac{1}{2}\right) & \frac{1}{2} \le t < \frac{3}{4}, \\ 2B\left(\frac{1}{4}\right) - B\left(\frac{1}{2}\right) & \frac{3}{4} \le t < 1, \end{cases} \tag{34}$$

$$\int_0^t \phi_3(s)dB(s) = \begin{cases} 0 & 0 \le t < \frac{1}{2} \\ B(t) - B\left(\frac{1}{2}\right) & \frac{1}{2} \le t < \frac{3}{4} \\ 2B\left(\frac{3}{4}\right) - B\left(\frac{1}{2}\right) - B(t) & \frac{3}{4} \le t < 1 \end{cases}$$

$$\simeq \begin{cases} 0 & 0 \le t < \dfrac{1}{2}, \\ B\left(\dfrac{5}{8}\right) - B\left(\dfrac{1}{2}\right) & \dfrac{1}{2} \le t < \dfrac{3}{4}, \\ 2B\left(\dfrac{3}{4}\right) - B\left(\dfrac{1}{2}\right) - B\left(\dfrac{7}{8}\right) & \dfrac{3}{4} \le t < 1. \end{cases} \tag{35}$$

We can rewrite (32)-(35), in terms of the RHFs $\phi_0(t)$, $\phi_1(t)$, $\phi_2(t)$ and $\phi_3(t)$, as follows:

$$\int_0^t \Phi(s)dB(s) \simeq S\Phi(t), \tag{36}$$

where 4×4 stochastic operational matrix of integration is given by

$$S_{4\times 4}$$

$$= \frac{1}{4}\begin{pmatrix} \alpha_{11} & \beta_{21} & 2\beta_{41} & 2\beta_{43} \\ \beta_{21} + 4B\left(\frac{1}{2}\right) & \alpha_{11} - 4B\left(\frac{1}{2}\right) & 2\beta_{41} & -2\beta_{43} \\ \beta_{41} - 2B\left(\frac{1}{2}\right) + 6B\left(\frac{1}{4}\right) & \beta_{41} + 2B\left(\frac{1}{2}\right) - 2B\left(\frac{1}{4}\right) & 2\alpha_{21} - 4B\left(\frac{1}{4}\right) & 0 \\ \beta_{43} - 2B\left(\frac{1}{2}\right) + 2B\left(\frac{3}{4}\right) & -\beta_{43} + 2B\left(\frac{1}{2}\right) - 2B\left(\frac{3}{4}\right) & 0 & 2\alpha_{22} - 4B\left(\frac{3}{4}\right) \end{pmatrix}. \tag{37}$$

In general, we have

$$S_{k\times k} = \frac{1}{k}\begin{pmatrix} C_{\left(\frac{k}{2}\right)\times\left(\frac{k}{2}\right)} & U_{\left(\frac{k}{2}\right)\times\left(\frac{k}{2}\right)} \\ V_{\left(\frac{k}{2}\right)\times\left(\frac{k}{2}\right)} & \overline{D}_{\left(\frac{k}{2}\right)\times\left(\frac{k}{2}\right)} \end{pmatrix}, \tag{38}$$

where

$$U_{\left(\frac{k}{2}\right)\times\left(\frac{k}{2}\right)} = \frac{k}{2}\hat{\Phi}_{\left(\frac{k}{2}\right)\times\left(\frac{k}{2}\right)} \cdot diag(\beta_{k1}, \beta_{k3}, \beta_{k5}, \ldots, \beta_{k,k-1}),$$

$$\overline{D}_{\left(\frac{k}{2}\right)\times\left(\frac{k}{2}\right)}$$

$$= \frac{k}{2} diag\left(\alpha_{\frac{k}{2},1} - 2B\left(\frac{1}{k}\right), \alpha_{\frac{k}{2},2} - 2B\left(\frac{3}{k}\right), \alpha_{\frac{k}{2},3} - 2B\left(\frac{5}{k}\right), ..., \alpha_{\frac{k}{2},\frac{k}{2}} - 2B\left(\frac{k-1}{k}\right)\right)$$

and

$$C_{1\times1} = \alpha_{11}, \quad C_{\left(\frac{k}{2}\right)\times\left(\frac{k}{2}\right)} = \frac{k}{2}\hat{S}_{\left(\frac{k}{2}\right)\times\left(\frac{k}{2}\right)},$$

where $\hat{S}_{\left(\frac{k}{2}\right)\times\left(\frac{k}{2}\right)}$ is the same matrix $S_{\left(\frac{k}{2}\right)\times\left(\frac{k}{2}\right)}$, with this difference that all coefficients $B\left(\frac{2i-1}{2p}\right)$, for $p = 1, 2, 2^2, 2^3, ..., \frac{k}{2}$, $i = 1, 2, 3, ..., p$ are doubled. Namely, for $k = 2, 4$, we have

$$C_{2\times2} = 2\hat{S}_{2\times2} = \begin{pmatrix} \alpha_{11} & \beta_{21} \\ \beta_{21} + 4B\left(\frac{1}{2}\right) & \alpha_{11} - 4B\left(\frac{1}{2}\right) \end{pmatrix},$$

$$C_{4\times4} = 4\hat{S}_{4\times4}$$

$$= \begin{pmatrix} \alpha_{11} & \beta_{21} & 2\beta_{41} & 2\beta_{43} \\ \beta_{21} + 8B\left(\frac{1}{2}\right) & \alpha_{11} - 8B\left(\frac{1}{2}\right) & 2\beta_{41} & -2\beta_{43} \\ \beta_{41} - 4B\left(\frac{1}{2}\right) + 12B\left(\frac{1}{4}\right) & \beta_{41} + 4B\left(\frac{1}{2}\right) - 4B\left(\frac{1}{4}\right) & 2\alpha_{21} - 8B\left(\frac{1}{4}\right) & 0 \\ \beta_{43} - 4B\left(\frac{1}{2}\right) + 4B\left(\frac{3}{4}\right) & -\beta_{43} + 4B\left(\frac{1}{2}\right) - 4B\left(\frac{3}{4}\right) & 0 & 2\alpha_{22} - 8B\left(\frac{3}{4}\right) \end{pmatrix},$$

and so

$$V_{1\times1} = \beta_{21} + 2B\left(\frac{1}{2}\right),$$

$$V_{2\times2} = \begin{pmatrix} \beta_{41} - 2B\left(\frac{1}{2}\right) + 6B\left(\frac{1}{4}\right) & \beta_{41} + 2B\left(\frac{1}{2}\right) + 2B\left(\frac{1}{4}\right) \\ \beta_{43} - 2B\left(\frac{1}{2}\right) + 2B\left(\frac{3}{4}\right) & -\beta_{43} + 2B\left(\frac{1}{2}\right) + 6B\left(\frac{3}{4}\right) \end{pmatrix},$$

$$V_{4\times 4}$$

$$= \begin{pmatrix} \beta_{81} - 6B\left(\frac{1}{4}\right) + 14B\left(\frac{1}{8}\right) & \beta_{81} + 2B\left(\frac{1}{4}\right) + 2B\left(\frac{1}{8}\right) & 2\beta_{81} + 4B\left(\frac{1}{4}\right) - 4B\left(\frac{1}{8}\right) & 0 \\ \beta_{83} - 6B\left(\frac{1}{4}\right) + 10B\left(\frac{3}{8}\right) - 4B\left(\frac{1}{2}\right) & \beta_{83} + 2B\left(\frac{1}{4}\right) - 6B\left(\frac{3}{8}\right) - 4B\left(\frac{1}{2}\right) & -2\beta_{83} + 4B\left(\frac{1}{4}\right) - 4B\left(\frac{3}{8}\right) & 0 \\ \beta_{85} - 4B\left(\frac{1}{2}\right) + 6B\left(\frac{5}{8}\right) - 2B\left(\frac{3}{4}\right) & -\beta_{85} + 4B\left(\frac{1}{2}\right) - 6B\left(\frac{5}{8}\right) + 2B\left(\frac{3}{4}\right) & 0 & 2\beta_{85} - 4B\left(\frac{5}{8}\right) + 4B\left(\frac{3}{4}\right) \\ \beta_{87} - 2B\left(\frac{3}{4}\right) + 2B\left(\frac{7}{8}\right) & -\beta_{87} + 2B\left(\frac{3}{4}\right) - 2B\left(\frac{7}{8}\right) & 0 & -2\beta_{87} + 4B\left(\frac{3}{4}\right) - 4B\left(\frac{7}{8}\right) \end{pmatrix}.$$

So, the Itô integral of every function $f(t)$ can be approximated as follows:

$$\int_0^t f(s)dB(s) \simeq \int_0^t F^T \Phi(s)dB(s) \simeq F^T S\Phi(t). \tag{39}$$

4. Solving m-dimensional Stochastic Itô-Volterra Integral Equations by Using Stochastic Operational Matrix

Consider the following linear stochastic Itô-Volterra integral equation with several independent white noise sources:

$$X(t) = f(t) + \int_0^t b(t, s)X(s)ds + \sum_{i=1}^{m} \int_0^t \sigma_i(t, s)X(s)dB_i(s), \quad t \in [0, 1). \tag{40}$$

Our problem is to determine block pulse coefficient of $X(t)$, where $X(t), f(t), b(t, s)$ and $\sigma_i(t, s)$, $i = 1, 2, \ldots, m$, for $t, s \in [0, 1)$, are the stochastic processes defined on the same probability space (Ω, F, P). Also, $B(t) = (B_1(t), B_2(t), \ldots, B_m(t))$ is m-dimensional Brownian motion and $\int_0^t \sigma_i(t, s)X(s)dB_i(s)$, $i = 1, 2, \ldots, m$, are the Itô integrals.

By using (6) and (9), we have consecutive approximations as:

$$X(t) \simeq X^T\Phi(t) = \Phi^T(t)X,$$

$$f(t) \simeq F^T\Phi(t) = \Phi^T(t)F,$$

$$b(t,\, s) \simeq \Phi^T(t)B\Phi(s) = \Phi^T(s)B^T\Phi(t),$$

$$\sigma_i(t,\, s) \simeq \Phi^T(t)\Sigma_i\Phi(s) = \Phi^T(s)\Sigma_i^T\Phi(t), \quad i = 1,\, 2,\, \ldots,\, m.$$

In the above approximations, X and F are the RHFs coefficient stochastic vectors, and B and Σ_i, $i = 1, 2, \ldots, m$ are the RHFs coefficient stochastic matrices.

With substituting above approximations in equation (40), we get

$$X^T\Phi(t) \simeq F^T\Phi(t) + X^T\left(\int_0^t \Phi(s)\Phi^T(s)ds\right)B^T\Phi(t)$$

$$+ X^T\left(\sum_{i=1}^{m}\left(\int_0^t \Phi(s)\Phi^T(s)dB_i(s)\right)\Sigma_i^T\right)\Phi(t). \tag{41}$$

Let b_j be the jth column of the constant matrix B, μ_{ij} be the jth column of the constant matrix Σ_i, p_i be the ith row of the integration operational matrix P, and s_{ij} be the jth row of the stochastic integration operational matrix S_i.

To illustrate the calculation procedures, we choose $\alpha = 1$ or $k = 4$. Using equations (15), (18)-(20) and (27), we get

$$\left(\int_0^t \Phi(s)\Phi^T(s)dB_i(s)\right)\Sigma_i^T\Phi(t) = \left(\int_0^t \Psi_{4\times 4}(s)dB_i(s)\right)\Sigma_i^T\Phi(t)$$

$$= \begin{pmatrix} s_{i1}\Phi(t) & s_{i2}\Phi(t) & s_{i3}\Phi(t) & s_{i4}\Phi(t) \\ s_{i2}\Phi(t) & s_{i1}\Phi(t) & s_{i3}\Phi(t) & -s_{i4}\Phi(t) \\ s_{i3}\Phi(t) & s_{i3}\Phi(t) & \dfrac{s_{i1}+s_{i2}}{2}\Phi(t) & 0 \\ s_{i4}\Phi(t) & -s_{i4}\Phi(t) & 0 & \dfrac{s_{i1}-s_{i2}}{2}\Phi(t) \end{pmatrix}\begin{pmatrix} \mu_{i1}^T\Phi(t) \\ \mu_{i2}^T\Phi(t) \\ \mu_{i3}^T\Phi(t) \\ \mu_{i4}^T\Phi(t) \end{pmatrix}$$

$$= \begin{pmatrix} s_{i1}\Phi(t)\Phi^T(t)\mu_{i1} + s_{i2}\Phi(t)\Phi^T(t)\mu_{i2} + s_{i3}\Phi(t)\Phi^T(t)\mu_{i3} + s_{i4}\Phi(t)\Phi^T(t)\mu_{i4} \\ s_{i2}\Phi(t)\Phi^T(t)\mu_{i1} + s_{i1}\Phi(t)\Phi^T(t)\mu_{i2} + s_{i3}\Phi(t)\Phi^T(t)\mu_{i3} - s_{i4}\Phi(t)\Phi^T(t)\mu_{i4} \\ s_{i3}\Phi(t)\Phi^T(t)\mu_{i1} + s_{i3}\Phi(t)\Phi^T(t)\mu_{i2} + \frac{s_{i1}+s_{i2}}{2}\Phi(t)\Phi^T(t)\mu_{i3} \\ s_{i4}\Phi(t)\Phi^T(t)\mu_{i1} - s_{i4}\Phi(t)\Phi^T(t)\mu_{i2} + \frac{s_{i1}+s_{i2}}{2}\Phi(t)\Phi^T(t)\mu_{i4} \end{pmatrix}$$

$$= \begin{pmatrix} s_{i1}\hat{\Gamma}_{i1} + s_{i2}\hat{\Gamma}_{i2} + s_{i3}\hat{\Gamma}_{i3} + s_{i4}\hat{\Gamma}_{i4} \\ s_{i2}\hat{\Gamma}_{i1} + s_{i1}\hat{\Gamma}_{i2} + s_{i3}\hat{\Gamma}_{i3} - s_{i4}\hat{\Gamma}_{i4} \\ s_{i3}\hat{\Gamma}_{i1} + s_{i3}\hat{\Gamma}_{i2} + \frac{s_{i1}+s_{i2}}{2}\hat{\Gamma}_{i3} \\ s_{i4}\hat{\Gamma}_{i1} - s_{i4}\hat{\Gamma}_{i2} + \frac{s_{i1}-s_{i2}}{2}\hat{\Gamma}_{i4} \end{pmatrix}\Phi(t)$$

$$= \begin{pmatrix} s_{i1} & s_{i2} & s_{i3} & s_{i4} \\ s_{i2} & s_{i1} & s_{i3} & -s_{i4} \\ s_{i3} & s_{i3} & \frac{s_{i1}+s_{i2}}{2} & 0 \\ s_{i4} & -s_{i4} & 0 & \frac{s_{i1}-s_{i2}}{2} \end{pmatrix}\begin{pmatrix} \hat{\Gamma}_{i1} \\ \hat{\Gamma}_{i2} \\ \hat{\Gamma}_{i3} \\ \hat{\Gamma}_{i4} \end{pmatrix}\Phi(t)$$

$$= (\widetilde{S}_i)_{4\times 4}(\widetilde{\Gamma}_i)_{4\times 1}\Phi(t) = (\hat{E}_i)_{4\times 4}\Phi(t), \quad i = 1, 2, ..., m.$$

In general, we have

$$\left(\int_0^t \Phi(s)\Phi^T(s)dB_i(s)\right)\Sigma_i^T\Phi(t) = (\hat{E}_i)_{k\times k}\Phi(t), \quad i = 1, 2, ..., m, \tag{42}$$

where

$$(\hat{E}_i)_{k\times k} = (\widetilde{S}_i)_{k\times k}(\widetilde{\Gamma}_i)_{k\times 1},$$

with

$$(\widetilde{S}_i)_{k\times k} = \begin{pmatrix} (\widetilde{S}_i)_{\left(\frac{k}{2}\right)\times\left(\frac{k}{2}\right)} & (M_i)_{\left(\frac{k}{2}\right)\times\left(\frac{k}{2}\right)} \\ (M_i^T)_{\left(\frac{k}{2}\right)\times\left(\frac{k}{2}\right)} & (\Lambda_i)_{\left(\frac{k}{2}\right)\times\left(\frac{k}{2}\right)} \end{pmatrix}, \tag{43}$$

where

$$(\widetilde{S}_i)_{1\times 1} = s_{i1},$$

$$(M_i)_{\left(\frac{k}{2}\right)\times\left(\frac{k}{2}\right)} = \hat{\Phi}_{\left(\frac{k}{2}\right)\times\left(\frac{k}{2}\right)} \cdot diag(s_{i,\frac{k}{2}+1}, s_{i,\frac{k}{2}+2}, ..., s_{ik}),$$

$$(\Lambda_i)_{\left(\frac{k}{2}\right)\times\left(\frac{k}{2}\right)} = diag[\hat{\Phi}^{-1}_{\left(\frac{k}{2}\right)\times\left(\frac{k}{2}\right)} \cdot (s_{i1}, s_{i2}, ..., s_{i,\frac{k}{2}})^T]^T$$

and

$$(\widetilde{\Gamma}_i)_{k\times 1} = \begin{pmatrix} \hat{\Gamma}_{i1} \\ \hat{\Gamma}_{i2} \\ \vdots \\ \hat{\Gamma}_{ik} \end{pmatrix}.$$

Similarly,

$$\left(\int_0^t \Phi(s)\Phi^T(s)ds\right)B^T\Phi(t) = E_{k\times k}\Phi(t), \tag{44}$$

where

$$E_{k\times k} = \widetilde{P}_{k\times k}\widetilde{B}_{k\times 1},$$

with

$$\widetilde{P}_{k\times k} = \begin{pmatrix} \widetilde{P}_{\left(\frac{k}{2}\right)\times\left(\frac{k}{2}\right)} & \widetilde{H}_{\left(\frac{k}{2}\right)\times\left(\frac{k}{2}\right)} \\ \widetilde{H}^T_{\left(\frac{k}{2}\right)\times\left(\frac{k}{2}\right)} & \Omega_{\left(\frac{k}{2}\right)\times\left(\frac{k}{2}\right)} \end{pmatrix}, \tag{45}$$

where

$$\widetilde{P}_{1\times 1} = p_1,$$

$$\widetilde{H}_{\left(\frac{k}{2}\right)\times\left(\frac{k}{2}\right)} = \hat{\Phi}_{\left(\frac{k}{2}\right)\times\left(\frac{k}{2}\right)} \cdot diag(p_{\frac{k}{2}+1}, p_{\frac{k}{2}+2}, ..., p_k),$$

$$\Omega_{\left(\frac{k}{2}\right)\times\left(\frac{k}{2}\right)} = diag[\hat{\Phi}^{-1}_{\left(\frac{k}{2}\right)\times\left(\frac{k}{2}\right)} \cdot (p_1, p_2, \ldots, p_{\frac{k}{2}})^T]^T$$

and

$$\widetilde{B}_{k\times 1} = \begin{pmatrix} \hat{B}_1 \\ \hat{B}_2 \\ \vdots \\ \hat{B}_k \end{pmatrix}.$$

With substituting relations (42) and (44) in (41), we get

$$X^T\Phi(t) \simeq F^T\Phi(t) + X^T E\Phi(t) + X^T\left(\sum_{i=1}^{m} \hat{E}_i\right)\Phi(t).$$

Then

$$X^T\left(I - E - \sum_{i=1}^{m} \hat{E}_i\right) \simeq F^T. \tag{46}$$

So, by setting $N = \left(I - E - \sum_{i=1}^{m} \hat{E}_i\right)^T$ and replacing $\simeq$ by =, we will have

$$NX = F \tag{47}$$

which is a linear system of equations that give the approximate RH functions coefficient of the unknown stochastic processes $X(t)$, so

$$X(t) \simeq X^T\Phi(t).$$

5. Numerical Examples in the Population Growth

Consider the following simple population growth model:

$$\begin{cases} \dfrac{dN(t)}{dt} = a(t)N(t), \\ N(0) = N_0, \end{cases} \tag{48}$$

where $N(t)$ is the number of population individuals at time t and $N(0) = N_0$ is the initial number at time $t = 0,$ and $a(t)$ is the growth rate at time t.

Suppose that $a(t)$ depends on several independent random environment effects, i.e.,

$$a(t) = \alpha(t) + \frac{1}{2}\sum_{i=1}^{m}\alpha_i^2(t) + \sum_{i=1}^{m}\beta_i(t)W_i(t),$$

where $W(t) = (W_1(t), W_2(t), \ldots, W_m(t))$ is an m-dimensional white noise with $W_i(t) = \frac{dB_i(t)}{dt}$, $i = 1, 2, \ldots, m$ and $B(t) = (B_1(t), B_2(t), \ldots, B_m(t))$ is an m-dimensional Brownian motion and so $\beta_i(t)$ is a nonrandom function that represented the infirmity and intensity of random environment impress of source i on noise term at time t, and $\alpha(t)$ is nonrandom growth relative rate at time t, and nonrandom functions $\alpha_i(t)$ represented the error of estimation of growth rate affected by random environment source i at time t.

Then we can define generalized stochastic exponential model of population growth as:

$$\begin{cases} \dfrac{dN(t)}{dt} = \left\{\alpha(t) + \dfrac{1}{2}\sum_{i=1}^{m}\alpha_i^2(t) + \sum_{i=1}^{m}\beta_i(t)W_i(t)\right\}N(t), \\ N(0) = N_0, \end{cases} \tag{49}$$

or

$$N(t) = N_0 + \int_0^t\left(\alpha(s) + \frac{1}{2}\sum_{i=1}^{m}\alpha_i^2(s)\right)N(s)ds + \sum_{i=1}^{m}\int_0^t\beta_i(s)N(s)dB_i(s). \tag{50}$$

The exact solution of stochastic differential equation (49) or stochastic integral equation (50) is given by:

$$N(t)=N_0\exp\left(\int_0^t\left[\alpha(s)+\frac{1}{2}\sum_{i=1}^{m}(\alpha_i^2(s)-\beta_i^2(s))\right]ds+\sum_{i=1}^{m}\int_0^t\beta_i(s)dB_i(s)\right). \quad (51)$$

Example 1. Consider the generalized stochastic exponential population growth model (50) for:

$$N_0 = 3\times 10^6,\ \alpha(t)=\sqrt{t},\ \alpha_1(t)=\frac{1}{10},\ \alpha_2(t)=\frac{1}{2},\ \alpha_3(t)=\frac{2}{5},$$

$$\beta_1(t)=\frac{t}{10},\ \beta_2(t)=\frac{\sin(t)}{8},\ \beta_3(t)=\frac{\cos(t)}{5}.$$

By using (51), the exact population is

$$N(t)=3\times 10^6\exp\left(\frac{2}{3}t^{\frac{3}{2}}-\frac{t^3}{600}+\frac{251}{1280}t-\frac{39}{12800}\sin(2t)+\frac{1}{10}\int_0^t sdB_1(s)\right.$$

$$\left.+\frac{1}{8}\int_0^t\sin(s)dB_2(s)+\frac{1}{5}\int_0^t\cos(s)dB_3(s)\right).$$

The approximate population for $k = 8, 16$ and exact population are shown in Table 1.

Table 1. The approximate population for $k = 8, 16$ and exact population

t	$k=8$	Exact population	$k=16$	Exact population
0	3044364	3000000	2712845	3000000
0.1	3044364	3182066	2915067	3036580
0.2	3244082	3356961	3535026	2825879
0.3	3420381	3568060	3398110	3174003
0.4	3503931	3333178	3263357	3493842
0.5	4230020	3729508	4363434	4478632
0.6	4230020	5188192	5493520	5040425
0.7	4184674	4396767	4383276	5184639
0.8	4437845	4719844	6166777	5948840
0.9	6802696	6521209	6697355	6380061
1	6802696	6562872	9605935	8807838
Relative error	0.080275		0.100782	

Example 2. Consider the generalized stochastic exponential population growth model (50) for:

$$N_0 = 5 \times 10^7,\ \alpha(t) = \ln(1+t),\ \alpha_1(t) = \frac{1}{50},\ \alpha_2(t) = \frac{3}{50},\ \alpha_3(t) = \frac{7}{50},$$

$$\beta_1(t) = \exp(-5t),\ \beta_2(t) = \frac{1}{(10+t)^3},\ \beta_3(t) = \frac{1}{150}\sin(t).$$

By using (51), the exact population is

$$N(t) = 5 \times 10^7 \exp\Bigg((1+t)\ln(1+t) + \frac{\exp(-10t)}{20} + \frac{1}{10(10+t)^5} + \frac{\sin(2t)}{180000}$$

$$- \frac{88939}{9000}t - \frac{1}{20} - 10^{-6} + \int_0^t \exp(-5s)\,dB_1(s)$$

$$+ \int_0^t \frac{1}{(10+s)^3}\,dB_2(s) + \frac{1}{150}\int_0^t \sin(s)\,dB_3(s)\Bigg).$$

The approximate population for $k = 8, 16$ and exact population are shown in Table 2.

Table 2. The approximate population for $k = 8, 16$ and exact population

t	$k = 8$	Exact population	$k = 16$	Exact population
0	51765796	50000000	38839020	50000000
0.1	51765796	44527970	48264041	38307964
0.2	51450931	52931623	48450386	53689898
0.3	43049507	41949173	57691495	49328192
0.4	42614948	41819435	54382754	54455818
0.5	63213233	52506494	48783808	48160566
0.6	63213233	66535138	52598742	51980972
0.7	66563238	57786764	51769125	48517420
0.8	69235074	59706259	57290571	46441019
0.9	76096164	68674659	61102214	69455533
1	76096164	66585983	62138305	67828762
Relative error	0.109092		0.128459	

6. Conclusion

Because it is almost impossible to find the exact solution of equation (40), it would be convenient to determine its numerical solution based on stochastic numerical analysis. Using rationalized Haar functions as basis functions to solve the linear stochastic Itô-Volterra integral equations with several independent white noise sources is very simple and effective in comparison with other methods. Its applicability and accuracy is checked on some examples. Moreover, one could also apply the Itô-Taylor expansion described by Kloeden and Platen [3], or those from article [30], for example. Certainly, it could be the topic of some future work.

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