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DIRICHLET PROBLEM IN A SIMPLY CONNECTED DOMAIN, BOUNDED BY THE NONTRIVIAL KIND

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Abstract

In this work, Dirichlet problem (Problem D) is considered for the regular solutions $u(x, y)$ of the elliptic differential equation of the second order $E(u) = 0$ with analytic coefficients in a flat domain T, bounded by an algebraic curve of genus $\rho = 0$.

1. Introduction

In this work, the consideration in [1] is formally retained, but the effectiveness of the method is conditional.

Consider the second order equation of the elliptic type as follows:

$$E(u) = \Delta u + a(x, y)\frac{\partial u}{\partial x} + b(x, y)\frac{\partial u}{\partial y} + c(x, y)u(x, y) = 0, \tag{1}$$

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where a, b and c are real analytic coefficients in a simply connected domain T. Analytic continuation of the coefficients into complex variables $z = x + iy$, $\zeta = x - iy$ yields:

$$F(U) = \frac{\partial^2 U}{\partial z \partial \zeta} + A(z, \zeta)\frac{\partial U}{\partial z} + B(z, \zeta)\frac{\partial U}{\partial \zeta} + C(z, \zeta)U(z, \zeta) = 0. \quad (2)$$

As the domain T is simply connected, the solution of Dirichlet problem (problem D) may be determined by following formulas:

$$u(x, y) = \operatorname{Re} U(z, \bar{z}), \quad (3)$$

$$U(z, \bar{z}) = G(z, \overline{z_0}; z, \bar{z})\phi_0(z) - \int_{z_0}^{z} \phi_0(t) H(z, \overline{z_0}; z, \bar{z})dt, \quad (4)$$

where $\phi_0(z)$ is a holomorphic function in T, chosen to satisfy the condition

$$u^+(t) = f(t),$$

or

$$\operatorname{Re} U(z, \zeta)\big|_{L_z} = \operatorname{Re} U(t, \bar{t})\big|_{t \in L} = f(t).$$

2. Solution Steps

1°. In the first step in problem solving, a complication arises when transforming the domain T onto Riemann surface $\Re_z$. In this case, the line L_z on $\Re_z$ will not be the boundary of one-sheeted simply connected domain T_z. The fact that the only oval $L : P(x, y) = 0$ for $\rho = 0$ is formed by all branches of this algebraic function, so that L consists of $\rho + 1$ smooth arcs contacting together L_j, transformation related group γ of line L into itself (or, anyway, the transformation group, which connects the branches of algebraic function $P(x, y) = 0$). In the transition $z = x + iy$, $\zeta = x - iy$ on Riemann surface

$$\Re_z : \rho\left(\frac{z + \zeta}{2}, \frac{z - \zeta}{2i}\right) = 0$$

the images L_{jz} arcs L_j, are adjoined and will lie on different sheets $\Re_z$, so that, L_z is the union of all L_{jz} $(j = 1, ..., \rho + 1)$.

To determine the boundary of the domain T_z, it is possible with the help of the considerations described in [2]. By replacements $z = x + iy$, $\zeta = x - iy$, the transformation group γ passes to group Γ and the transformation of surface $\Re_z$ into itself. Together with the line L_z on the surface $\Re_z$ all lines will be lines of symmetry, Γ-congruent to L_z. Each sheet of the arc Γ-congruent lines formed closed contour, projection onto the plane z is line L. The interior of any of these contours can take T_z on $\Re_z$.

Therefore, the boundary of T_z consists of arc L_z, that lies on one sheet $\Re_z$, and ρ the arcs of other lines of symmetry $\Re_z$, Γ-congruent L_z.

Here is a complete analogy with the case of arc polygons on the plane. The domain T_z is a half of fundamental domain of the group Γ on $\Re_z$. Its second half T_z^* is symmetric with T_z relative to one of its sides. For example, relative to arc L_{1z} with the known law of symmetry $(z, \zeta) \to (\overline{\zeta}, \overline{z})$. Γ-the congruent forms of domain $T_z \cup T_z^* \cup L_{1z}$ form parquet surface coverage $\Re_z$.

2°. On the basis of Section 1° the second step in solving a problem D, consists of constructing Schwarz's operator $U(z, \zeta)$ for the domain T_z on the surface $\Re_z$ with the boundary condition

$$\operatorname{Re} U\big|_{\partial T_z} = f(t), \tag{5}$$

Since T_z is a symmetrical halves of fundamental domain groups Γ on $\Re_z$, the construction of $U(z, \zeta)$ can use the symmetric method in the form, as it is described in [5]. Auxiliary function $\Omega(z, \zeta)$ introduced by equalities

$$\Omega(z, \zeta) = \begin{cases} U(z, \zeta), \ (z, \zeta) \in T_z, \\ -\overline{U(\overline{\zeta}, \overline{z})}, \ (z, \zeta) \in T_z^*, \end{cases}$$

extends over the entire surface $\Re_z$ by automorphy relative to transformation group Γ:

$$\Omega(z, \zeta) = \Omega(z_k, \zeta_k), \quad (z_k, \zeta_k) \in \Gamma, \quad k = 1, 2, \ldots \tag{6}$$

and on the basis of boundary condition (5) obtained on the surface $\Re_z$ a special case of the Riemann problem-jump problem with boundary condition

$$\Omega^+(t, \bar{t}) - \Omega^-(t, \bar{t}) = 2f(t), \quad t \in L. \tag{7}$$

Its solutions must be holomorphic on $\Re_z$ for all $(z, \zeta) \notin \partial T_z$, integrated with $(t, \bar{t}) \in T_z$, Γ-automorphic and satisfying the symmetry condition

$$\Omega(z, \zeta) = \overline{-\Omega(\bar{\zeta}, \bar{z})}.$$

The scheme for solving the jump problem known from the analog of Cauchy kernel $\widetilde{K}(z, \zeta; \xi, \eta)$ for the surface $\Re_z$ by the formula

$$\widetilde{K}(z, \zeta; \xi, \eta) = \sum_{k=0}^{N} K(z_k, \zeta_k; \xi, \eta)$$

constructed an automorphic Cauchy kernel $\widetilde{K}(z, \zeta; \xi, \eta)$ and the solution is written in the integral form as

$$\Omega(z, \zeta) = \frac{1}{2\pi i}\int_L f(\tau)\{K(z, \zeta; \tau, \bar{\tau}) + K(\zeta, z; \tau, \bar{\tau})\}d\tau + iM_0, \tag{8}$$

where instead of K it becomes $\widetilde{K}$. However, we should take into account that for $\rho \neq 0$, the analog Cauchy's kernel $K(z, \zeta; \xi, \eta)$ has simple poles in ρ points $(z, \zeta) = (\alpha_k, \beta_k)$, $k = 1, \ldots, \rho$ surface $\Re_z$. Obviously, in these points, and in all their Γ-integral images, (8) can also have simple poles, and integral (8) holomorphic at all these points when and only when its residues at points $(\alpha_1, \beta_1), \ldots, (\alpha_\rho, \beta_\rho)$ become zero, i.e.,

$$res_{(\alpha_k, \beta_k)}\Omega(z, \zeta) = 0, \quad k = 1, \ldots, \rho. \tag{9}$$

It is known from [2] that equalities (9) are equivalent to the relationships

$$\int_L f(t)\theta_k(t, \bar{t})dt = 0, \quad k = 1, ..., \rho, \tag{10}$$

where the functions $\theta_1(z, \zeta), \theta_2(z, \zeta), ..., \theta_\rho(z, \zeta)$ represent the base of the Abelian differentials of the first kind of surface $\Re_z$. Using the conditions (10), the solution $\Omega(z, \zeta)$ of the jump problem (7) will satisfy all required conditions and the unknown integral of Schwarz $U(z, \zeta)$ for domain $T_z \subset \Re_z$ will be determined by known equality

$$U(z, \zeta) = \Omega(z, \zeta), \qquad (z, \zeta) \in T_z.$$

3°. The last step of calculations consists of rotation integral equation:

$$G(z, \overline{z_0}; z, w(z))\phi_0(z) - \int_{z_0}^{z} \phi_0(t)H(z, \overline{z_0}; z, w(z))dt = \Omega(z, w(z)),$$

in the cases $\rho = 0$ and $\rho \neq 0$ are identical. Therefore, the process for solving problem D in the case $\rho \neq 0$ is complete. Finally, we formulate a theorem stressed only for characteristic properties of this case noted in the points 1° and 2°.

Theorem 1. *The simply connected domain T, bounded by algebraic curve of genus* $\rho \neq 0$, *is a flat model of the Riemann surface of the same genus. Problem D for the elliptical equation* (1) *with the analytical coefficients for domain T has at least one solution when and only when boundary function* $f(t)$ *satisfies the additional conditions in the form* (10).

3. Test Problem

Consider the domain T, bounded by the curve

$$L : x^4 + y^4 = a^4, \quad a > 0. \tag{11}$$

This domain is called a *pseudo-square*. The boundary of this domain L is an algebraic curve of degree 4, which does not have double points as shown in Figure 1. It means that the genus of this curve [5] is

$$\rho = \frac{(n-1)(n-2)}{2} = \frac{3 \cdot 2}{2} = 3.$$

The number ρ can be computed also by another formula [5] as the genus of Riemann surface $\Re_z$. This surface has four sheets

$$y_k = e^{i\pi k/2} y, \quad y = \sqrt[4]{a^4 - x^4}, \quad k = 0, 1, 2, 3, \tag{12}$$

it has four branch points $(ae^{i\pi k/2}, 0)$, each of them is connected to all four sheets. Therefore by formula

$$\rho = \frac{1}{2}\sum_i (r_i - 1) - \nu + 1 = \frac{1}{2} \cdot 3 \cdot 4 - 4 + 1 = 3,$$

where $(r_i - 1)$ is the total branching of surface and ν is the number of sheets.

Let us construct for curve (11) surface symmetries $\Re_z$. From equation (11), it is obtained in the form:

$$\left(\frac{z+\zeta}{2}\right)^4 + \left(\frac{z-\zeta}{2i}\right)^4 = a^4$$

or, in the form

$$\Re_z : \zeta^4 + 6z^2\zeta^2 + z^4 - 8a^4 = 0. \tag{13}$$

It is four-sheeted and each sheet is a branch of algebraic function $\zeta = \zeta_k(z)$, $k = 1, 2, 3, 4$, determined by equation (13):

$$\zeta = \pm\sqrt{-3z^2 \pm \sqrt{8(z^4 + a^4)}}. \tag{14}$$

$\Re_z$ has two sets of the branch points of the second degree:

(1) 4 branch points in the internal square root on (14),

$$z_k = ae^{\frac{(2k+1)\pi i}{4}}, \quad k = 0, 1, 2, 3;$$

these are the roots of equation $z^4 + a^4 = 0$; each of z_k lies on two branch points $(z_k, \pm\widetilde{\zeta}_k)$, in this case by (14), $\widetilde{\zeta}_k^2 + 3z_k^2 = 0$;

(2) 4 branch points in the external square roots $(\widetilde{z}_l, 0)$, $l = 0, 1, 2, 3$, which $\widetilde{z}_l$ are the roots of the equation

$$-3z^2 \pm \sqrt{8(z^4 + a^4)} = 0 \Rightarrow z^4 = 8a^4$$

have the values

$$\widetilde{z}_l = a\sqrt[4]{8}e^{i\pi l/2} = \frac{2a}{\sqrt[4]{2}}e^{i\pi l/2}, \quad l = 0, 1, 2, 3.$$

To determine on $\Re_z$ the cuts between the branch points and its connection diagram sheets, let us apply the well-known method. All the points z_k and $\widetilde{z}_l$ on the complex plane z are connected by a simple piecewise-smooth line σ, that tends to infinity as is shown in Figure 2. Here σ begins from the point z_0, and passes through others in the sequence as follows:

$$z_0, \widetilde{z}_1, z_1, \widetilde{z}_2, z_2, \widetilde{z}_3, z_3, \widetilde{z}_0, \infty.$$

We assume that, the orientation on σ corresponds to these location points and if the plane z was cut at σ, then "+" side σ^+ will be on the left, and "–" side of σ^- on the right. To determine, how the sheets $\Re_z$ are connected by the line σ on the interval $[z_0, \widetilde{z}_1]$, it is necessary to fix some arrangement for sheets $\Re_z$ to the left of the interval $[z_0, \widetilde{z}_1]$ (we can say that on its left side $[z_0, \widetilde{z}_1]^+$) and then by going around the point z_0 go to its right side $[z_0, \widetilde{z}_1]^-$, making sure, in this case that the moving around changes branches $\zeta_j(z)$ of algebraic function (14).

By determining the connection diagrams of sheets $\Re_z$ on the interval $[\widetilde{z}_1, z_1] \subset \sigma$ transition σ^+ to σ^- corresponds around the two branch points z_0 and $\widetilde{z}_1$. If the interval $[z_1, \widetilde{z}_2] \subset \sigma$ is considered, then during the transfer of σ^+ to σ^-, it is necessary to bypass the three points $z_0, \widetilde{z}_1, z_1$.

As the characteristics of branches $\zeta_j(z)$ around the branch points are used, the substitution scheme becomes

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{pmatrix}.$$

We allow the comparison of numbers of two lines under each other to judge the transition from one branch to another. To summarize description of the study, we do not use the substitutions system in this example. We write the table

$$\begin{array}{lcc} \zeta_1(z) & +\,- & 1 \\ \zeta_2(z) & +\,+ & 2 \\ \zeta_3(z) & -\,+ & 3 \\ \zeta_4(z) & -\,- & 4 \end{array} \tag{15}$$

indicating that the branches $\zeta_j(z),\ \ j = 1, 2, 3, 4$ of the algebraic function (13) fixed somehow in the radicals (14) differ from each other by a combination of signs in front of these radicals and the sheets $\Re_z$ to bypass z_0 have numbers 1, 2, 3, 4 from top to bottom. In this case the sheet with the number j is filled with value branches $\zeta_j(z).$ From

$$\sqrt{z^4 + a^4} = \sqrt{(z - z_0)(z - z_1)(z - z_2)(z - z_3)},$$

it is evident that the circuit around any of the points z_k changes sign before the radical, and hence this approach in table (15) corresponds to a change of each sign of the second column. From the power series expansion in the neighborhood of any point

$$-3z^2 \pm \sqrt{8(z^4 + a^4)} = d_{1,l}^{\pm}(z - \tilde{z}_l) + d_{2,l}^{\pm}(z - \tilde{z}_l)^2 + \cdots,\ d_{1,l}^{\pm} \neq 0,$$

it follows that to circuit around any of the points $\tilde{z}_l$ in table (15) corresponds change in each sign of the first column. On this basis, the circuits around point z_0, two points z_0 and $\tilde{z}_1$, and three points z_0, $\tilde{z}_1$ and z_1 can be described by the corresponding columns of the signs, obtained from the columns of table (15) and the corresponding variable.

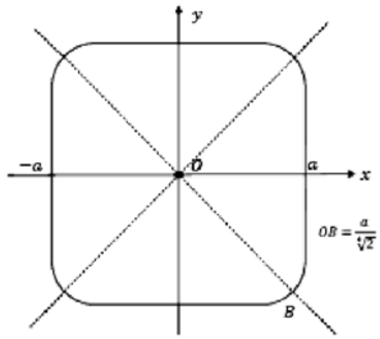

Figure 1

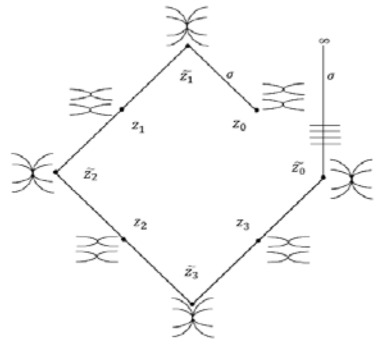

Figure 2

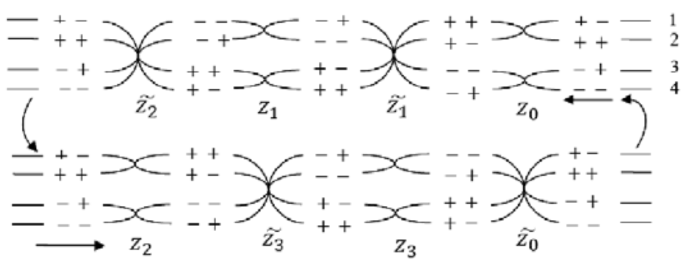

Figure 3

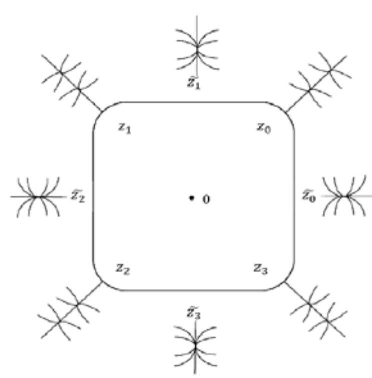

Figure 4

In Figure 3, we displayed all pairs of the columns of signs, which characterize entire system of the circuits of branch points. The first row must be bypassed from right to left, and the second-from left to right, in accordance with the position of branch points on σ. Below between each pair of the columns of signs is indicated the branch point, circuit around which was completed.

As soon as all identical pairs of signs in the different columns in Figure 3 were connected by continuous line, we get a graphical representation of scheme connections sheets $\Re_z$ over all branch points at the same time.

On the basis of obtained scheme we find that on $z = \infty$ there are no branch points. The same conclusion can be derived on the basis of the formula (14), which we obtain as

$$\lim_{z\to\infty} \frac{\zeta}{z} = \lim_{z\to\infty} \pm\sqrt{-3 \pm \sqrt{8 + \frac{8a^4}{z^4}}} = \pm\sqrt{-3 \pm \sqrt{8}}.$$

These four different limit values indicate, that on $z = \infty$ lie four different points.

Schema described in Figure 3 helps to choose the system of cuts between branch points in the model surface $\Re_z$. It is possible to save the cuts in a polygonal chain σ and when the coasts of cuts on all four sheets are connected together by the schema, we get four-sheeted surface $\Re_z$ by two lines of transition from sheet to sheet on the broken $[z_0, \tilde{z}_1, z_1, \tilde{z}_2]$ and $[z_2, \tilde{z}_3, z_3, \tilde{z}_0]$. The scheme of this model $\Re_z$ is shown in Figure 2.

In accordance with the same schema cuts on all sheets can be carried out on the rectilinear rays $[z_k, \infty]$ and $[\tilde{z}_l, \infty]$, $k, l = 0, 1, 2, 3$ connected by the already known schemes. This model $\Re_z$ is more symmetric and easily drawn in imagination. Its scheme is shown in Figure 4.

Now let us see the way the line L_z lies on $\Re_z$.

Using Figure 4, when the point z runs by the side of pseudo-square

$[z_3, z_0]$, the point $(z, \zeta_1(z))$ on the upper sheet $\Re_z$ describes the same arc, which lies above $[z_3, z_0] \subset L$. The side $[z_0, z_1] \subset L$ is obtained from $[z_3, z_0]$ on rotation by an angle $\pi/2$ around the origin. On the basis of adopted fix branches, we have

$$\zeta_1(iz) = \sqrt{3z^2 - \sqrt{8(z^4 + a^4)}} = i\sqrt{-3z^2 + \sqrt{8(z^4 + a^4)}} = i\zeta_2(z).$$

It means that point $(iz, \zeta_1(iz)) = (iz, \zeta_2(iz))$ describes the second sheet $\Re_z$ arc with the projection $[z_0, z_1] \subset L$. The remaining two sides $[z_1, z_2] \subset L$ and $[z_2, z_3]$ that turn at angles π and $3\pi/2$ are obtained to lie on $[z_0, z_1]$. But if we consider that to happen, then

$$\zeta_1(-z) = \zeta_1(z) = -\zeta_4(z),$$

$$\zeta_1(-iz) = \zeta_1(iz) = i\zeta_2(z) = -i\zeta_3(z).$$

Thus the points $(-z, \zeta_1(-z)) = (-z, -\zeta_4(-z))$ and $(-iz, \zeta_1(-iz)) = (-iz, -i\zeta_3(z))$ describe, respectively, arcs on the fourth and third sheets $\Re_z$, that lie over the sides of pseudo-square $[z_1, z_2]$ and $[z_2, z_3]$. As a result closed curve L_z^1, located on all four sheets is obtained to lie on $\Re_z$. This one of the symmetry lines on $\Re_z$, for which $\Re_z$ dissymmetric. If we trace the motion of points $(z, \zeta_2(z))$, $(z, \zeta_3(z))$ and $(z, \zeta_4(z))$ on $\Re_z$ by the same way with the complete circuit of the point z of the boundary of pseudo-square L, then we get three more closed lines L_z^2, L_z^3, L_z^4 each of which is also located on all the four sheet $\Re_z$ and is the line of symmetry $\Re_z$. Obviously, these four lines are obtained from one other by the birational transformations

$$(ze^{i\pi k/2}, we^{i\pi k/2}), \quad k = 0, 1, 2, 3.$$

Lines $L_z^1, L_z^2, L_z^3, L_z^4$, intersecting at the branch points, on each sheet cut out the domain in the form of a pseudo-square. Any of these domains can be taken to be an image T_z.

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