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SOME HADAMARD-TYPE INEQUALITIES ON FRACTIONAL INTEGRAL AND APPLICATIONS

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Abstract

In this paper, we present some new integral inequalities via Hadamard integral and apply these inequalities to construct special inequalities.

1. Introduction

In recent years, inequalities are playing a very significant role in all fields of mathematics, and have applications in many fields. Consider the functional

$$T(f, g) := \frac{1}{b-a}\int_a^b f(x)g(x)dx - \left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right), \quad (1.1)$$

where f and g are two integrable functions which are synchronous on $[a, b]$, (i.e., $((f(x) - f(y))(g(x) - g(y)) \geq 0$ for any $x, y \in [a, b]$), given in [1]. Many researchers have given considerable attention to (1.1) and number of inequalities appeared in literature see [2, 7, 8].

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Recently many authors have studied integral inequalities on fractional calculus using Riemann-Liouville, Caputo derivative, see [2-5] and the references therein.

Another kind of fractional derivative is the fractional derivative due to Hadamard [6]. Recently in the literature, there appeared some results on fractional integral inequalities using Hadamard fractional integral; see [7-10].

The aim of this paper is to establish two integral inequalities using Hadamard fractional integral.

2. Preliminaries

In this section, we give some preliminaries and basic proposition used in this paper. We give some definitions of Hadamard fractional integral as in [11, p. 159-171]. The necessary background details are given in the book by Kilbas et al. [12].

Definition 1. The *Hadamard fractional integral* of order $\alpha \in \mathbb{R}^+$ of a function $f(t)$, for all $t > 1$ is defined as

$$ {}_H D_{1,t}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{x} \right)^{\alpha - 1} f(x) \frac{dx}{x}, \tag{2.1}$$

where Γ is the gamma function defined by

$$\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha-1} dx,$$

where $\alpha > 0$. Note that $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$.

From the above definitions, we can see the difference between Hadamard fractional and Riemann-Liouville fractional integrals. The kernel in the Hadamard integral has the form of $\ln\left(\frac{t}{x}\right)$ instead of the form of $(x - t)$.

In [8], Chinchane and Pachpatte presented a fractional integral inequality via Hadamard integral as follows.

Theorem 2 [8]**.** *Let f and g be two functions on* $[0, \infty)$ *such that*

$$(f(x) - f(y))(g(x) - g(y)) \geq 0$$

for all x, y. Then

$${}_H D_{1,t}^{-\alpha}(fg)(t) \geq \frac{\Gamma(\alpha+1)}{(\ln t)^{\alpha}}\left({}_H D_{1,t}^{-\alpha} f(t)\right)\left({}_H D_{1,t}^{-\alpha} g(t)\right)$$

for all $\alpha > 0,\ t > 1.$

In [9], Sroysang presented new inequalities on Hadamard fractional integral as follows.

Theorem 3 [9]**.** *Let f, g and h be functions on* $[0, \infty)$ *such that*

$$(f(x) - f(y))(g(x) - g(y))(h(x) + h(y)) \geq 0$$

for all x, y. Then

$${}_H D_{1,t}^{-\alpha}(fgh)(t)\,{}_H D_{1,t}^{-\alpha}(t) + {}_H D_{1,t}^{-\alpha}(fg)(t)\,{}_H D_{1,t}^{-\alpha} h(t)$$

$$\geq {}_H D_{1,t}^{-\alpha} g(t)\,{}_H D_{1,t}^{-\alpha}(fh)(t) + {}_H D_{1,t}^{-\alpha} f(t)\,{}_H D_{1,t}^{-\alpha}(gh)(t)$$

for all $\alpha > 0,\ t > 1.$

3. Results

Theorem 4. *Let f, g, h and k be functions on* $[0, \infty)$ *such that*

$$(f(x) - f(y))(g(x) - g(y))(h(x) + h(y))(k(x) + k(y)) \geq 0$$

for all x, y. Then

$${}_H D_{1,t}^{-\alpha}(fghk)(t)\,{}_H D_{1,t}^{-\alpha}(1) + {}_H D_{1,t}^{-\alpha}(fgh)(t)\,{}_H D_{1,t}^{-\alpha} k(t)$$

$$+ {}_H D_{1,t}^{-\alpha}(fgk)(t)\,{}_H D_{1,t}^{-\alpha} h(t) + {}_H D_{1,t}^{-\alpha}(fg)(t)\,{}_H D_{1,t}^{-\alpha}(hk)(t)$$

$$\geq {}_H D_{1,t}^{-\alpha} f(t)\,{}_H D_{1,t}^{-\alpha}(ghk)(t) + {}_H D_{1,t}^{-\alpha} g(t)\,{}_H D_{1,t}^{-\alpha}(fhk)(t)$$

$$+ {}_H D_{1,t}^{-\alpha}(fh)(t)\,{}_H D_{1,t}^{-\alpha}(gk)(t) + {}_H D_{1,t}^{-\alpha}(fk)(t)\,{}_H D_{1,t}^{-\alpha}(gh)(t)$$

for all $\alpha > 0,\ t > 1.$

Proof. By the assumption, for any x, y, we have

$$(fghk)(x) + (fghk)(y) + (fgh)(x)k(y) + k(x)(fgh)(y)$$

$$+ (fgk)(x)h(y) + h(x)(fgk)(y) + (fg)(x)(hk)(y) + (hk)(x)(fg)(y)$$

$$\geq f(x)(ghk)(y) + (ghk)(x)f(y) + g(x)(fhk)(y) + (fhk)(x)g(y)$$

$$+ (fh)(x)(gk)(y) + (gk)(x)(fh)(y) + (fk)(x)(gh)(y)$$

$$+ (gh)(x)(fk)(y).$$

Then

$$\int_1^t \left(\ln\frac{t}{x}\right)^{\alpha-1} (fghk)(x)\frac{dx}{x} + \int_1^t \left(\ln\frac{t}{x}\right)^{\alpha-1} (fghk)(y)\frac{dx}{x}$$

$$+ \int_1^t \left(\ln\frac{t}{x}\right)^{\alpha-1} (fgh)(x)k(y)\frac{dx}{x}$$

$$+ \int_1^t \left(\ln\frac{t}{x}\right)^{\alpha-1} k(x)(fgh)(y)\frac{dx}{x} + \int_1^t \left(\ln\frac{t}{x}\right)^{\alpha-1} (fgk)(x)h(y)\frac{dx}{x}$$

$$+ \int_1^t \left(\ln\frac{t}{x}\right)^{\alpha-1} h(x)(fgk)(y)\frac{dx}{x}$$

$$+ \int_1^t \left(\ln\frac{t}{x}\right)^{\alpha-1} (fg)(x)(hk)(y)\frac{dx}{x} + \int_1^t \left(\ln\frac{t}{x}\right)^{\alpha-1} (hk)(x)(fg)(y)\frac{dx}{x}$$

$$\geq \int_1^t \left(\ln\frac{t}{x}\right)^{\alpha-1} f(x)(ghk)(y)\frac{dx}{x} + \int_1^t \left(\ln\frac{t}{x}\right)^{\alpha-1} (ghk)(x)f(y)\frac{dx}{x}$$

$$+ \int_1^t \left(\ln\frac{t}{x}\right)^{\alpha-1} g(x)(fhk)(y)\frac{dx}{x}$$

$$+ \int_1^t \left(\ln\frac{t}{x}\right)^{\alpha-1} (fhk)(x)g(y)\frac{dx}{x} + \int_1^t \left(\ln\frac{t}{x}\right)^{\alpha-1} (fh)(x)(gk)(y)\frac{dx}{x}$$

$$+\int_1^t\left(\ln\frac{t}{x}\right)^{\alpha-1}(gk)(x)(fh)(y)\frac{dx}{x}$$

$$+\int_1^t\left(\ln\frac{t}{x}\right)^{\alpha-1}(fk)(x)(gh)(y)\frac{dx}{x}+\int_1^t\left(\ln\frac{t}{x}\right)^{\alpha-1}(gh)(x)(fk)(y)\frac{dx}{x}.$$

Consequently,

$$\begin{aligned}
&{}_HD_{1,t}^{-\alpha}(fghk)(t)+(fghk)(y)_HD_{1,t}^{-\alpha}(1)+k(y)_HD_{1,t}^{-\alpha}(fgh)(t)\\
&+(fgh)(y)_HD_{1,t}^{-\alpha}(k)(t)+(h)(y)_HD_{1,t}^{-\alpha}(fgk)(t)\\
&+(fgk)(y)_HD_{1,t}^{-\alpha}(h)(t)+(hk)(y)_HD_{1,t}^{-\alpha}(fg)(t)\\
&+(fg)(y)_HD_{1,t}^{-\alpha}(hk)(t)\\
\geq\ &(ghk)(y)_HD_{1,t}^{-\alpha}(f)(t)+f(y)_HD_{1,t}^{-\alpha}(ghk)(t)\\
&+(fhk)(y)_HD_{1,t}^{-\alpha}g(t)+g(y)_HD_{1,t}^{-\alpha}(fhk)(t)\\
&+(gk)(y)_HD_{1,t}^{-\alpha}(fh)(t)+(fh)(y)_HD_{1,t}^{-\alpha}(gk)(t)\\
&+(gh)(y)_HD_{1,t}^{-\alpha}(fk)(t)+(fk)(y)_HD_{1,t}^{-\alpha}(gh)(t),
\end{aligned}$$

where $\alpha>0$, $t>1$ and $y\in(1,t)$.

Similarly, we can write

$$\begin{aligned}
&{}_HD_{1,t}^{-\alpha}(fghk)(t)_HD_{1,t}^{-\alpha}(1)+{}_HD_{1,t}^{-\alpha}(fgh)(t)_HD_{1,t}^{-\alpha}k(t)\\
&+{}_HD_{1,t}^{-\alpha}(fgk)(t)_HD_{1,t}^{-\alpha}h(t)+{}_HD_{1,t}^{-\alpha}(fg)(t)_HD_{1,t}^{-\alpha}(hk)(t)\\
\geq\ &{}_HD_{1,t}^{-\alpha}f(t)_HD_{1,t}^{-\alpha}(ghk)(t)+{}_HD_{1,t}^{-\alpha}g(t)_HD_{1,t}^{-\alpha}(fhk)(t)\\
&+{}_HD_{1,t}^{-\alpha}(fh)(t)_HD_{1,t}^{-\alpha}(gk)(t)+{}_HD_{1,t}^{-\alpha}(fk)(t)_HD_{1,t}^{-\alpha}(gh)(t),
\end{aligned}$$

where $\alpha>0$, $t>1$ and this ends the proof.

Theorem 5. *Let f, g, h and k be functions on* $[0, \infty)$ *such that*

$$(f(x) - f(y))(g(x) - g(y))(h(x) - h(y))(k(x) - k(y)) \geq 0$$

for all x, y. Then

$$\begin{aligned} & {}_H D_{1,t}^{-\alpha}(fg)(t)\,{}_H D_{1,t}^{-\alpha}(hk)(t) + {}_H D_{1,t}^{-\alpha}(gk)(t)\,{}_H D_{1,t}^{-\alpha}(fh)(t) \\ & + {}_H D_{1,t}^{-\alpha}(fk)(t)\,{}_H D_{1,t}^{-\alpha}(gh)(t) + {}_H D_{1,t}^{-\alpha}(fghk)(t)\,{}_H D_{1,t}^{-\alpha}(1) \\ \geq\ & {}_H D_{1,t}^{-\alpha}(fhk)(t)\,{}_H D_{1,t}^{-\alpha}(g)(t) + {}_H D_{1,t}^{-\alpha}(fgk)(t)\,{}_H D_{1,t}^{-\alpha}(h)(t) \\ & + {}_H D_{1,t}^{-\alpha}(k)(t)\,{}_H D_{1,t}^{-\alpha}(fgh)(t) + {}_H D_{1,t}^{-\alpha}(f)(t)\,{}_H D_{1,t}^{-\alpha}(ghk)(t) \end{aligned}$$

for all $\alpha > 0,\ t > 1.$

Proof. By the assumption, for any x, y, we have

$$\begin{aligned} & (fg)(x)(hk)(y) + (hk)(x)(fg)(y) + (gk)(x)(fh)(y) \\ & + (fk)(x)(gh)(y) + (gh)(x)(fk)(y) \\ & + (fh)(x)(gk)(y) + (fghk)(x) + (fghk)(y) \\ \geq\ & (fhk)(x)g(y) + (fgk)(x)h(y) + k(x)(fgh)(y) \\ & + (fgh)(x)k(y) + h(x)(fgk)(y) \\ & + (ghk)(x)f(y) + g(x)(fhk)(y) + f(x)(ghk)(y). \end{aligned}$$

Similar to the proof of Theorem 4, we have

$$\begin{aligned} & (hk)(y)\,{}_H D_{1,t}^{-\alpha}(fg)(t) + (fg)(y)\,{}_H D_{1,t}^{-\alpha}(hk)(t) \\ & + (fh)(y)\,{}_H D_{1,t}^{-\alpha}(gk)(t) + (gh)(y)\,{}_H D_{1,t}^{-\alpha}(fk)(t) \\ & + (fk)(y)\,{}_H D_{1,t}^{-\alpha}(gh)(t) + (gk)(y)\,{}_H D_{1,t}^{-\alpha}(fh)(t) \\ & + {}_H D_{1,t}^{-\alpha}(fghk)(t) + (fghk)(y)\,{}_H D_{1,t}^{-\alpha}(1) \end{aligned}$$

$$\geq (g)(y)_H D_{1,t}^{-\alpha}(fhk)(t) + (h)(y)_H D_{1,t}^{-\alpha}(fgk)(t)$$

$$+ (fgh)(y)_H D_{1,t}^{-\alpha}(k)(t) + (k)(y)_H D_{1,t}^{-\alpha}(fgh)(t)$$

$$+ (fgk)(y)_H D_{1,t}^{-\alpha}(h)(t) + (f)(y)_H D_{1,t}^{-\alpha}(ghk)(t)$$

$$+ (fhk)(y)_H D_{1,t}^{-\alpha}(g)(t) + (ghk)(y)_H D_{1,t}^{-\alpha}(f)(t).$$

Similar to the proof of Theorem 4, we can write

$$_H D_{1,t}^{-\alpha}(fg)(t)_H D_{1,t}^{-\alpha}(hk)(t) + {}_H D_{1,t}^{-\alpha}(gk)(t)_H D_{1,t}^{-\alpha}(fh)(t)$$

$$+ {}_H D_{1,t}^{-\alpha}(fk)(t)_H D_{1,t}^{-\alpha}(gh)(t) + {}_H D_{1,t}^{-\alpha}(fghk)(t)_H D_{1,t}^{-\alpha}(1)$$

$$\geq {}_H D_{1,t}^{-\alpha}(fhk)(t)_H D_{1,t}^{-\alpha}(g)(t) + {}_H D_{1,t}^{-\alpha}(fgk)(t)_H D_{1,t}^{-\alpha}(h)(t)$$

$$+ {}_H D_{1,t}^{-\alpha}(k)(t)_H D_{1,t}^{-\alpha}(fgh)(t) + {}_H D_{1,t}^{-\alpha}(f)(t)_H D_{1,t}^{-\alpha}(ghk)(t),$$

where $\alpha > 0,\ t > 1$ and this ends the proof.

4. Applications

Now using the results of Section 3, we give some special inequalities.

Example 1. The assertion follows from Theorem 4 applied for $f(x) = g(x) = h(x) = k(x) = x$ on $[0, \infty)$ and $\alpha = 1$.

Under the assumptions Theorem 4, we have inequality,

$$(x - y)(x - y)(x + y)(x + y) \geq 0,$$

$$x^4 + y^4 \geq 2x^2y^2.$$

Then

$$\frac{1}{4}(t^4 - 1) \cdot \frac{\ln t}{\Gamma(2)} + \frac{1}{4}(t^4 - 1) \cdot \frac{\ln t}{\Gamma(2)} \geq 2 \cdot \frac{1}{2}(t^2 - 1) \cdot \frac{1}{2}(t^2 - 1),$$

$$\ln t \geq \frac{t^2 - 1}{t^2 + 1}.$$

Example 2. The assertion follows from Theorem 4 applied for $f(x) = g(x) = h(x) = k(x) = x$ on $[0, \infty)$ and $\alpha = 2$.

Under the assumptions Theorem 4, we have inequality

$$(x - y)(x - y)(x + y)(x + y) \geq 0,$$

$$x^4 + y^4 \geq 2x^2y^2.$$

Then

$$\frac{1}{4}\left[\frac{1}{4}(t^4 - 1) - \ln t\right] \cdot \frac{(\ln t)^2}{\Gamma(3)} + \frac{1}{4}\left[\frac{1}{4}(t^4 - 1) - \ln t\right] \cdot \frac{(\ln t)^2}{\Gamma(3)}$$

$$\geq 2 \cdot \frac{1}{2}\left[\frac{1}{2}(t^2 - 1) - \ln t\right] \cdot \frac{1}{2}\left[\frac{1}{2}(t^2 - 1) - \ln t\right],$$

$$(\ln t)^2 \geq \frac{2[(t^2 - 1) - 2\ln t]^2}{[(t^4 - 1) - 4\ln t]}.$$

Example 3. The assertion follows from Theorem 4 applied for $f(x) = x + c_1$, $g(x) = xe^x + c_2$, $h(x) = x$, $k(x) = xe^x$ on $[0, \infty)$ and $\alpha = 1$.

Under the assumptions Theorem 4, we have inequality

$$(x^2 - y^2)(x^2e^{2x} - y^2e^{2y}) \geq 0,$$

$$x^4e^{2x} + y^4e^{2y} \geq x^2y^2e^{2x} + x^2y^2e^{2y}.$$

Then

$$2 \cdot \left(\frac{1}{2}t^3e^{2t} - \frac{3}{4}t^2e^{2t} + \frac{3}{4}te^{2t} - \frac{3}{8}e^{2t} - \frac{1}{8}e^2\right) \cdot \ln t$$

$$\geq 2 \cdot \left[\frac{1}{2}(t^2 - 1) \cdot \left(\frac{1}{2}te^{2t} - \frac{1}{4}e^{2t} - \frac{1}{4}e^2\right)\right],$$

$$\ln t \geq \frac{(t^2 - 1) \cdot ((2t - 1)e^{2t} - e^2)}{(4t^3 - 6t^2 + 6t - 3)e^{2t} - e^2}.$$

Example 4. The assertion follows from Theorem 5 applied for $f(x) = g(x) = h(x) = k(x) = x$ on $[0, \infty)$ and $\alpha = 1$.

Under the assumptions Theorem 5, we have inequality

$$(x - y)^4 \geq 0,$$

$$x^4 + 6x^2y^2 + y^4 \geq 4x^3y + 4xy^3.$$

Then

$$\frac{1}{4}(t^4 - 1) \cdot \frac{\ln t}{\Gamma(2)} + 6 \cdot \frac{1}{2}(t^2 - 1)\frac{1}{2}(t^2 - 1) + \frac{1}{4}(t^4 - 1) \cdot \frac{\ln t}{\Gamma(2)}$$

$$\geq 4 \cdot \frac{1}{3}(t^3 - 1)(t - 1) + 4 \cdot (t - 1) \cdot \frac{1}{3}(t^3 - 1),$$

$$\ln t \geq \frac{(t - 1)(7t^2 - 2t + 7)}{3(t + 1)(t^2 + 1)}.$$

Example 5. The assertion follows from Theorem 5 applied for $f(x) = x$, $g(x) = x^2$, $h(x) = x^3$, $k(x) = x^4$ on $[0, \infty)$ and $\alpha = 1$.

Under the assumptions Theorem 5, we have inequality

$$(x - y)(x^2 - y^2)(x^3 - y^3)(x^4 - y^4) \geq 0,$$

$$x^{10} + 2x^5y^5 + y^{10} \geq xy^9 + x^2y^8 + x^8y^2 + x^9y.$$

Then

$$\frac{1}{10}(t^{10} - 1) \cdot \frac{\ln t}{\Gamma(2)} + 2 \cdot \frac{1}{5}(t^5 - 1)\frac{1}{5}(t^5 - 1) + \frac{1}{10}(t^{10} - 1) \cdot \frac{\ln t}{\Gamma(2)}$$

$$\geq (t - 1) \cdot \frac{1}{9}(t^9 - 1) + \frac{1}{2}(t^2 - 1) \cdot \frac{1}{8}(t^8 - 1) + \frac{1}{8}(t^8 - 1) \cdot \frac{1}{2}(t^2 - 1)$$

$$+ \frac{1}{9}(t^9 - 1) \cdot (t - 1),$$

$$\ln t \geq \frac{400(t-1)(t^9-1)+225(t^2-1)(t^8-1)-144(t^5-1)^2}{360(t^{10}-1)}.$$

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