**Far East Journal of Mathematical Sciences (FJMS)**

Published Online: August 2016
http://dx.doi.org/10.17654/MS100050783
Volume 100, Number 5, 2016, Pages 783-794 **ISSN: 0972-0871**

# ON SUPER WHEEL-ANTIMAGIC TOTAL LABELING FOR A WHEEL $k$-MULTILEVEL CORONA WITH A CYCLE

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## Abstract

A simple graph $G(V, E)$ admits an $(a, d)$-$H$-antimagic total labeling if every edge in $E(G)$ belongs to at least one subgraph of $G$ isomorphic to $H$ and there exists a bijection $f: V(G) \cup E(G) \to \{1, 2, ..., |V(G)| + |E(G)|\}$ such that for all subgraphs $H'$ isomorphic to $H$, the set of $H$-weights, $w(H') = \sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e)$ constitutes an arithmetic progression $a, a + d, ..., a + (t - 1)d$, where $a$ and $d$ are some positive integers and $t$ is

Received: April 14, 2016; Revised: May 3, 2016; Accepted: May 28, 2016
2010 Mathematics Subject Classification: 05C78.
Keywords and phrases: wheel-antimagic, $k$-multilevel corona.
Communicated by K. K. Azad

the number of subgraphs of $G$ isomorphic to $H$. Moreover, if $f(V(G)) = \{1, 2, ..., |V(G)|\}$, then $G$ is called a super $(a, d)$-$H$-antimagic. This work aims at to study super $(a, d)$-$H$-antimagic total labeling on corona of a wheel $k$-multilevel corona with a cycle.

## 1. Introduction

The concept of $H$-magic labeling was first introduced by Gutiérrez and Lladó [2] in 2005, which generalized the concept of magic valuation originated by Rosa [10]. Let $H$ and $G(V, E)$ be finite simple graphs with every edge of $G$ in at least one subgraph isomorphic to $H$. A bijection $f : V(G) \cup E(G) \to \{1, 2, ..., V(G)| + |E(G)|\}$ is an $H$-magic total labeling of $G$ if there exists a positive integer $m(f)$ called the *magic sum* such that for any subgraph $H'(V', E')$ of $G$ isomorphic to $H$, the sum $\sum_{v \in V'} f(v) + \sum_{e \in E'} f(e)$ equals to $m(f)$. When $f(v)_{v \in V} = \{1, 2, ..., |V|\}$, we say that $G$ is $H$-*supermagic*.

Many results on $H$-magic total labelings of a graph have been found. Gutiérrez and Lladó [2] proved that a star $S_n$ and a complete bipartite $K_{m,n}$ are both $S_h$-supermagic, and a path $P_n$ and a cycle are $P_h$-supermagic. Lladó and Moragas [6] found that some graphs such as wheels, windmills, and books are $C_n$-magic graphs. Maryati et al. [8] showed that some classes of trees are path $P_h$-supermagic. Jeyanthi and Selvagopal [5] proved that shackles and amalgamations are $H$-supermagic for some $H$. In [11], Roswitha and Baskoro showed that a caterpillar, a double star, a firecracker and a banana tree graph admit $H$-supermagic. Recently, Marbun and Salman [7] have found that a wheel corona $k$-multilevel with a cycle is a wheel-supermagic.

In 2009, Inayah et al. [3] introduced $(a, d)$-$H$-antimagic total labeling by combining $H$-magic total labeling and $(a, d)$-edge-antimagic total labeling. Suppose that $G$ admits $H$-total labeling. Then an $(a, d)$-$H$-

antimagic total labeling is a bijective function $f : V(G) \cup E(G) \to \{1, 2, ..., |V(G)| + |E(G)|\}$ such that for all subgraphs $H'$ isomorphic to $H$, the set of $H$-weights, $w(H') = \sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e)$ constitutes an arithmetic progression $a, a + d, ..., a + (t - 1)d$, where $a$ and $d$ are some positive integers and $t$ is the number of subgraphs of $G$ isomorphic to $H$. Inayah et al. [3] proved that a fan $F_n$ admits an $(a, d)$-cycle $C_n$-antimagic total labeling for some $d$ and in [4] showed that shackles of a connected graph $H$ are super $(a, d)$-$H$-antimagic. Recent results are found in Roswitha et al. (see [12] and [13]).

In this paper, we find a super $(a, d)$-$H$-antimagic covering on a wheel $k$-multilevel corona with a cycle $W_n \odot C_n$, where $H = W_n$. Yero et al. [14] defined that graph $G$ corona graph $H$, denoted by $G \odot H$, is a graph obtained from $G$ and $|V(G)|$ copies of $H$, namely $H_1, H_2, ..., H_{|V(G)|}$, then joined every $v_i \in V(G)$ to all vertices in $V(H_i)$. Let $k$ be a positive integer. Graph $G$ $k$-multilevel corona with graph $H$, denoted by $G \odot^k H$, is a graph defined by $(G \odot^{k-1} H) \odot H$, for $k \geq 2$.

## 2. Technique of Partitioning a Multiset

A multiset is a set that allows the existence of same elements in it (Maryati et al. [9]). Let $X$ be a set containing some positive integers. We use the notation $[a, b]$ to mean $\{x \in \mathbb{N} | a \leq x \leq b\}$ and $\Sigma X$ to mean $\sum_{x \in X} x$. For any $k \in \mathbb{N}$, the notation $k + [a, b]$ means $\{k + x | x \in [a, b]\}$. According to Gutiérrez and Lladó [2], the set $X$ is $k$-equipartition if there exist $k$ subsets of $X$, say $X_1, X_2, ..., X_k$ such that $\bigcup_{i=1}^{k} X_i = X$ and $|X_i| = \frac{|X|}{k}$ for every $i \in [1, k]$.

The union operation of a multiset provides $\{1\} \uplus \{1, 2\} = \{1, 1, 2\}$. Let $Y$ be a multiset containing positive integers. Then $Y$ is said to be *k-balanced* if there exist $k$ subsets of $Y$, i.e., $Y_1, Y_2, \ldots, Y_k$ such that for every $i \in [1, k]$, $|Y_i| = \frac{|Y|}{k}$, $\Sigma Y_i = \frac{\Sigma Y}{k} \in \mathbb{N}$ and $\biguplus_{i=1}^{k} Y_i = Y$ [9]. In 2013, Inayah et al. [4] introduced $(k, \delta)$-anti balanced as follows. The multiset $Y$ is $(k, \delta)$-anti balanced if there exist $k$ subsets of $Y$, say $Y_1, Y_2, \ldots, Y_k$ such that for every $i \in [1, k]$, $|Y_i| = \frac{|Y|}{k}$, $\biguplus_{i=1}^{k} Y_i = Y$, and for $i \in [1, k-1]$, $\Sigma Y_{i+1} - \Sigma Y_i = \delta$ is satisfied. Then we have the following lemmas:

**Lemma 2.1.** *Let $k$ and $h$ be two positive integers and $k \geq 2$. If $X = [1, kh]$, then it is $(k, h)$-anti balanced.*

**Proof.** We arrange the set of integers $X = [1, kh]$ in a $k \times h$ matrix $A$ as given below:

$$A = \begin{pmatrix} 1 & k+1 & \cdots & (h-1)k+1 \\ 2 & k+2 & \cdots & (h-1)k+2 \\ \vdots & \vdots & & \vdots \\ k & 2k & \cdots & kh \end{pmatrix}.$$

Here $A = (a_{i,j})_{k \times h}$, where $a_{i,j} = k(j-1)+i$, for $1 \leq i \leq k$ and $1 \leq j \leq h$. Define $X_i = \sum_{j=1}^{h} a_{i,j}$. It can be verified that every subset $X_i$ has $h$ elements and $\biguplus_{i=1}^{k} X_i = X$. For every $1 \leq i \leq k$, the sum of all elements in $X_i$ is $\sum X_i = \sum_{j=1}^{h} X_i = \frac{1}{2} kh(h-1) + hi$, and $d = \sum X_{i+1} - \sum X_i = h$. □

**Lemma 2.2.** *Let $k$ and $h$ be positive integers and $k$ be even. If $X = [1, kh]$, then $X$ is $(k, k)$-anti balanced.*

**Proof.** Let $x_i^j$ be a $j$th-element of the subset $X_i$ which can be defined as

$$x_i^j = \begin{cases} i, & \text{for } j = 1, \\ \frac{k}{2}(k + j - 1) + i, & \text{for } i \in \left[1, \frac{k}{2}\right] \text{ and } j \in \left[2, \frac{k}{2}\right], \\ \frac{k}{2}(j - 1) + i, & \text{for } i \in \left[\frac{k}{2} + 1, k\right] \text{ and } j \in \left[2, \frac{k}{2}\right], \\ \frac{k}{2} j + i, & \text{for } i \in \left[1, \frac{k}{2}\right] \text{ and } j \in \left[\frac{k}{2} + 1, k\right], \\ \frac{k}{2}(k + j - 2) + i, & \text{for } i \in \left[\frac{k}{2} + 1, k\right] \text{ and } j \in \left[\frac{k}{2} + 1, k\right]. \end{cases}$$

Then the sum of all elements in $X_i$ for $i \in [1, k]$ is $\Sigma X_i = ki + \frac{k^2}{2} + \frac{k^3}{2}$. It is clear that $d = \Sigma X_{i+1} - \Sigma X_i = k$, then the multiset $X$ is $(k, k)$-anti balanced. □

## 3. Main Results

### 3.1. A wheel $k$-multilevel corona with a cycle $W_n \odot^k C_n$

A wheel $W_n$ is a graph obtained by joining $n$ vertices of $C_n$ with one central vertex (Bača and Miller [1]). Let $G$ be a graph $W_n \odot^k C_n$. Then $G$ consists of one wheel $W_n$ and $\sum_{i=1}^{k}(n+1)^i$ cycles named $C_j^i$ for $i \in [1, k]$ and $j \in [1, (n+1)^i]$. The order of $G$ is $|V(G)| = (n+1)^{k+1}$ and the size is $|E(G)| = 2n\sum_{i=0}^{k}(n+1)^i$.

**Theorem 3.1.** *Let $n \geq 3$ and $k \geq 2$ be positive integers. Then $G$ is a super $(a, d)$-$W_n$-antimagic total labeling for $d = 3n + 1$.*

**Proof.** Let $X = \left[1, (n+1)^{k+1} + 2n\sum_{i=0}^{k}(n+1)^i\right]$. Partition $X$ into two sets: $Y = [1, (n+1)^{k+1}]$ and $Z = (n+1)^{k+1} + \left[1, 2n\sum_{i=0}^{k}(n+1)^i\right]$. Now, we define a total labeling $f$ on $G$ as follows. We use the elements of $Y$ to label all the vertices of $W_n \odot^k C_n$ and the elements of $Z$ to label all the edges of $W_n \odot^k C_n$. Here, we have two cases to be considered.

**Case 1.** $n$ is even.

Let $Y$ be partitioned into $Y^l$, $1 \leq l \leq k+1$.

(1) For $l = 1$, we define $Y^l = [1, (n+1)^{k+1}]$.

Apply Lemma 2.1 to partition $Y^l$ into $(n+1)^k$ subsets such that $\sum Y_i^1 = \frac{1}{2}n(n+1)^{k+1} + (n+1)$, and $|\Sigma Y_i^1| = n+1$, for $i \in [1, (n+1)^k]$. Then it is obvious that $\Sigma Y_{i+1}^1 - \Sigma Y_i^1 = n+1$.

(2) For $2 \leq l < k+1$, we set that $Y^l = \frac{1}{2}(n+1)^{k-l}(n^2 + 2n + 2)$ $\sum_{i=1}^{l-1}(n+1)^i + [1, (n+1)^{k-l+2}]$. Then $Y^l$ can be partitioned into $(n+1)^{k-l+1}$ subsets. By using Lemma 2.1, we have $|Y_i^l| = n+1$ and $\sum Y_i^l = \sum_{j=2}^{l}(n+1)^{k+3-j} + \frac{1}{2}n(n+1)^{k+1} + (n+1)i$, for $\in [1, (n+1)^{k-l+1}]$. The difference between $\Sigma Y_{i+1}^l$ and $\Sigma Y_i^l$ is $n+1$ and we also have $\sum Y_1^l - \sum Y_{(n+1)^{k-l+2}}^{l-1} = n+1$.

(3) For $l = k+1$, let $Y^l = \frac{1}{2}(n+1)^{k-l}(n^2 + 2n + 2)\sum_{i=1}^{l-1}(n+1)^i +$ $[1, (n+1)^{k-l+2}]$. Then we have $\sum Y^{k+1} = \sum_{j=2}^{k+1}(n+1)^{k+3-j} + \frac{1}{2}n(n+1)^{k+1}$ $+(n+1)$ with $|Y^{k+1}| = n+1$, and hence $\Sigma Y^{k+1} - \Sigma Y_{n+1}^{k} = n+1$.

**Case 2.** $n$ is odd.

(1) For $l = 1$, we define and partition $Y^1$ similar to Case 1.

(2) For $k > 2$ and $2 \le l < k$, we also define and partition $Y^l$ as in Case 1.

(3) For $l = k$, let $Y^l = \frac{1}{2}(n+1)^{k-l}(n^2 + 2n + 2)\sum_{i=1}^{l-1}(n+1)^i + [1, (n+1)^{k-l+2}]$.

Now we use Lemma 2.2 to partition $Y^k$ into $n+1$ subsets such that $\sum Y_i^k = \sum_{j=2}^{k}(n+1)^{k+3-j} + \frac{1}{2}n(n+1)^{k+1} + (n+1)i$, for $i \in [1, n+1]$. This gives us $|Y^k| = n+1$. It is clear that $\Sigma Y_{i+1}^k - \Sigma Y_i^k = n+1$ and $\Sigma Y_1^k - \Sigma Y_{(n+1)^2}^{k-1} = n+1$.

(4) For $l = k+1$, let $Y^l = \{x_i^j \,|\, x_i^j \in Y^k\}$, where $j = \frac{n+3}{2}$ for $i \in [3, n+1]$ and $i = 1$, and $j = \frac{n+5}{2}$ for $i = 2$. We find that $|Y^{k+1}| = n+1$ and $\sum Y^{k+1} = \sum_{j=2}^{k+1}(n+1)^{k+3-j} + \frac{1}{2}n(n+1)^{k+1} + (n+1)$, and $\sum Y^{k+1} - \sum Y_{n+1}^k$ $n+1$.

Next, we use the elements of $Y^{k+1}$ to label the set of vertices in $W_n$. Then label the set of vertices in $C_j^1$ with the elements of set $Y^k \backslash Y^{k+1}$ for $j \in [1, n+1]$. Meanwhile, label the set of vertices in $C_j^i$ with the elements of set $Y^{k+1-i} \backslash Y^{k+2-i}$, for $j \in [1, (n+1)^i]$ and $i \in [2, k]$. We have $\sum_{i=0}^{k}(n+1)^i$ subgraphs of $W_n \odot^k C_k$ isomorphic to $W_n$. Let $h \in \left[1, \sum_{i=0}^{k}(n+1)^i\right]$. Then the vertex weight of $W_n^h$ is $f(V(W_n^h)) = \frac{1}{2}n(n+1)^{k+1} + (n+1)h$.

Furthermore, we partition $Z$ into $\sum_{i=0}^{k}(n+1)^i$ subsets according to Lemma 2.1 such that the sum of labels in $Z_i$ is

$$2in + (4n-1)((n+1)^{k+1}-1) + 2n \text{ for } h \in \left[1, \sum_{i=0}^{k}(n+1)^i\right].$$

Use all elements of $Z_i$ to label all the edges in $W_n^h$ by considering the order of $f(V(W_n^h))$. The $W_n^h$-weight is

$$w(W_n^h) = \frac{1}{2}n(n+1)^{k+1} + (n+1)h + 2hn + (4n-1)((n+1)^{k+1}-1) + 2n$$

$$= \frac{9n}{2}(n+1)^{k+1} - (n+1)^{k+1} + 3hn - 2n + h + 1.$$

We obtain that $d = w(W_n^{h+1}) - w(W_n^h) = 3n+1$, and $a = \frac{9n}{2}(n+1)^{k+1} - (n+1)^{k+1} + n + 2$. This completes the proof. □

**Corollary 3.2.** *Graph* $W_n \odot^k C_n$ *admits*

(1) *a super* $\left(\frac{9n}{2}(n+1)^{k+1} + 1,\ n+1\right)$-$W_n$*-antimagic*,

(2) *a super* $\left(\frac{5n}{2}(n+1)^{k+1} + 2n^2 + 2n + 1,\ 4n^2 + n + 1\right)$-$W_n$*-antimagic*.

Figure 3.1 illustrates $W_n \odot^k C_n$ for $n = 3$ and $k = 2$. Tables 3.1-3.4 show how the labels are partitioned according to Theorem 3.1.

**Table 3.1.** The partition of $Y_i^1$ on $W_3 \odot^2 C_3$

| $i$ | | | | | $\Sigma Y_i$ |
|---|---|---|---|---|---|
| 1 | 1 | 17 | 33 | 49 | 100 |
| 2 | 2 | 18 | 34 | 50 | 104 |
| 3 | 3 | 19 | 35 | 51 | 108 |
| 4 | 4 | 20 | 36 | 52 | 112 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| 15 | 15 | 31 | 47 | 63 | 156 |
| 16 | 16 | 32 | 48 | 64 | 160 |

**Table 3.2.** The partition of $Y_i^2$ on $W_3 \odot^2 C_3$

| $i$ | | | | | $\Sigma Y_i$ |
|---|---|---|---|---|---|
| 1 | 35 | 45 | 41 | 43 | 164 |
| 2 | 36 | 46 | 42 | 44 | 168 |
| 3 | 37 | 39 | 47 | 49 | 172 |
| 4 | 38 | 40 | 48 | 50 | 176 |

**Table 3.3.** The partition of $Y^3$ on $W_3 \odot^2 C_3$

| | | | | $\Sigma Y_i$ |
|---|---|---|---|---|
| 41 | 44 | 47 | 48 | 180 |

**Table 3.4.** The partition of $Z$ on $W_3 \odot^2 C_3$

| $i$ | | | | | | | $\Sigma Y_i$ |
|---|---|---|---|---|---|---|---|
| 1 | 65 | 86 | 107 | 128 | 149 | 170 | 705 |
| 2 | 66 | 87 | 108 | 129 | 150 | 171 | 711 |
| 3 | 67 | 88 | 109 | 130 | 151 | 172 | 717 |
| 4 | 68 | 89 | 110 | 131 | 152 | 173 | 723 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | | | $\vdots$ |
| 20 | 84 | 105 | 126 | 147 | 168 | 189 | 819 |
| 21 | 85 | 106 | 127 | 148 | 169 | 190 | 825 |

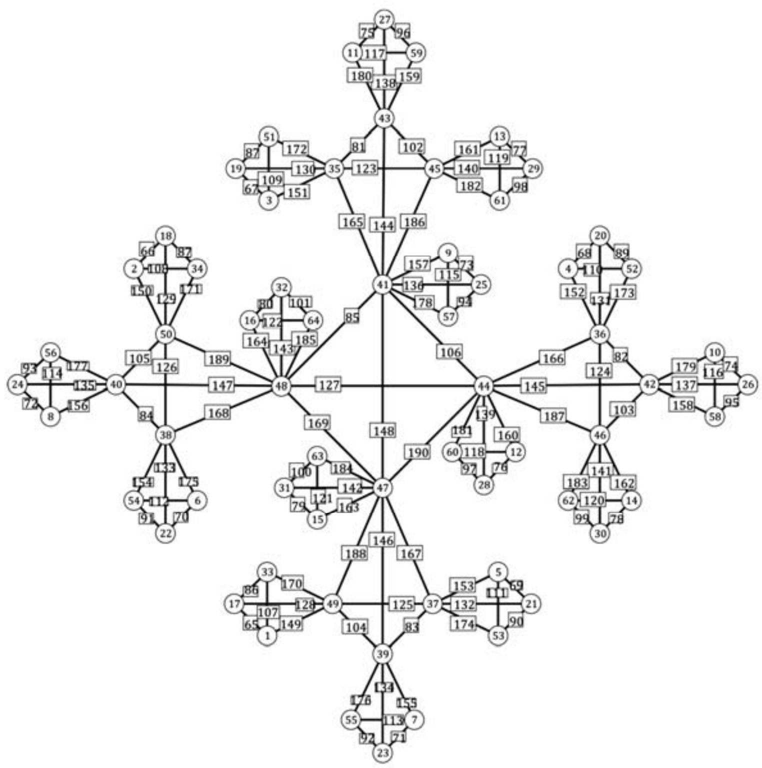

**Figure 3.1.** A super $(a, d)$-wheel-antimagic total labeling on $W_3 \odot^2 C_3$ with $d = 3n + 1$.

**Acknowledgements**

The authors would like to thank DIKTI Indonesia for the funding and support of this research under research grant PUPT 2016.

The authors thank the anonymous referees for their valuable suggestions which led to the improvement of the manuscript.

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