Far East Journal of Mathematical Sciences (FJMS)

Published Online: August 2016
http://dx.doi.org/10.17654/MS100050763
Volume 100, Number 5, 2016, Pages 763-771 **ISSN: 0972-0871**

THE INTERSECTION PROPERTY OF QUASI-IDEALS IN RINGS OF STRICTLY UPPER TRIANGULAR MATRICES OVER A EUCLIDEAN DOMAIN

Ronnason Chinram

Algebra and Applications Research Unit
Department of Mathematics and Statistics
Faculty of Science
Prince of Songkla University
Hat Yai, Songkhla 90110
Thailand
e-mail: ronnason.c@psu.ac.th

Centre of Excellence in Mathematics
CHE, Si Ayuthaya Road
Bangkok 10400, Thailand

Abstract

The notion of quasi-ideals for rings was introduced by Steinfeld. The intersection of a left ideal and a right ideal of R is a quasi-ideal of R but a quasi-ideal of R need not be obtained in this way. A quasi-ideal Q of R is said to have the intersection property if Q is the intersection of a left ideal and a right ideal of R. If every quasi-ideal of R has the intersection property, then R is said to have the intersection property of quasi-ideals. Let $SU_n(R)$ be the ring of all strictly upper triangular

Received: April 1, 2016; Accepted: May 31, 2016

2010 Mathematics Subject Classification: 16S50.

Keywords and phrases: ring of strictly upper triangular matrices, intersection property of quasi-ideals, Euclidean domain.

Communicated by K. K. Azad

$n \times n$ matrices over a ring R. Kemprasit and Juntarakhajorn [3] shows that if F is a field, then the ring $SU_n(F)$ has the intersection property of quasi-ideals if and only if $n \leq 3$. In this paper, we provide a characterization, when the ring $SU_n(R)$ has the intersection property of quasi-ideals in terms of n, where R is a Euclidean domain.

1. Introduction and Preliminaries

For nonempty subsets A, B of a ring R, let $\mathbb{Z}A$ and AB denote, respectively, the set of all finite sums of the form $\sum k_i a_i$, where $k_i \in \mathbb{Z}$ and $a_i \in A$ and the set of all finite sums of the form $\sum a_i b_i$, where $a_i \in A$ and $b_i \in B$. A subring Q of R is called a *quasi-ideal* of R if $RQ \cap QR \subseteq Q$. The notion of quasi-ideals for rings was introduced by Steinfeld in [5]. Quasi-ideals generalize left ideals and right ideals. For a nonempty subset X of a ring R, let $(X)_q$ be the quasi-ideal of R generated by X which is the intersection of all quasi-ideals of R containing X [6, p. 11]. In [7], Weinert has given the next proposition.

Proposition 1.1 [7]**.** *For a nonempty subset X of a ring R*,

$$(X)_q = \mathbb{Z}X + (RX \cap XR).$$

It is well-known that the intersection of a left ideal and a right ideal of R is a quasi-ideal of R but a quasi-ideal of R need not be obtained in this way. We can see some examples in [6, p. 8] and [1, 3, 4, 8]. We say that a quasi-ideal Q of R *has the intersection property* if Q is the intersection of a left ideal and a right ideal of R. Then every left ideal and every right ideal of R has the intersection property. If every quasi-ideal of R has the intersection property, then R is said to *have the intersection property of quasi-ideals*. Commutative rings, rings having a one-sided identity and regular rings are the examples of rings having the intersection property of quasi-ideals ([6], p. 9 and p. 73, respectively). In [8], Zhang et al. have characterized the following two known results:

Proposition 1.2 [8]**.** *Let X be a nonempty subset of a ring R. Then the following statements are equivalent*:

(i) $(X)_q$ *has the intersection property*.

(ii) $(\mathbb{Z}X + RX) \cap (\mathbb{Z}X + XR) = (X)_q$.

(iii) $RX \cap (\mathbb{Z}X + XR) \subseteq (X)_q$.

(iv) $XR \cap (\mathbb{Z}X + RX) \subseteq (X)_q$.

Proposition 1.3 [8]**.** *A ring R has the intersection property of quasi-ideals if and only if for any finite nonempty subset X of R*,

$$RX \cap (\mathbb{Z}X + XR) \subseteq \mathbb{Z}X + (RX \cap XR).$$

Let $SU_n(R)$ be the ring of all strictly upper triangular $n \times n$ matrices over a ring *R*. In [3], Kemprasit and Juntarakhajorn have characterized when $SU_n(F)$, where *F* is a field has the intersection property of quasi-ideals as follows:

Proposition 1.4 [3]**.** *Let F be a field. Then the ring* $SU_n(F)$ *has the intersection property of quasi-ideals if and only if* $n \leq 3$.

The purpose of this paper is to give the necessary and sufficient conditions for *n* that the ring $SU_n(R)$ has the intersection property of quasi-ideals where *R* is a Euclidean domain.

2. Main Result

Let *R* be a Euclidean domain. Then *R* is a PID and also a GCD domain.

Lemma 2.1. *Let R be a Euclidean domain and* $a, b, c \in R \setminus \{0\}$. *Then the following statements hold*:

(i) $aR + bR = \gcd(a, b)R$.

(ii) $aR \cap bR = lcm(a, b)R$.

(iii) $\gcd(a, lcm(b, c)) \mid lcm(\gcd(a, b), c)$.

(iv) $(aR + bR) \cap cR \subseteq aR + (bR \cap cR)$.

Proof. (i) and (ii) are clear.

(iii) Since R is a GCD domain, we have that $ab = \gcd(a, b)lcm(a, b)$. We can show that $\gcd(a, lcm(b, c)) \mid lcm(\gcd(a, b), c)$.

(iv) By (i) and (ii), we have that

$$(aR + bR) \cap cR = \gcd(a, b)R \cap cR = lcm(\gcd(a, b), c)R$$

and

$$aR + (bR \cap cR) = aR + lcm(b, c)R = \gcd(a, lcm(b, c))R.$$

By (iii), we have $\gcd(a, lcm(b, c)) \mid lcm(\gcd(a, b), c)$, this implies that $lcm(\gcd(a, b), c)R \subseteq \gcd(a, lcm(b, c))R$. Hence, $(aR + bR) \cap cR \subseteq aR + (bR \cap cR)$, as required. □

Lemma 2.2. *Let R be a Euclidean domain such that every additive subgroup is an ideal of R. The ring $SU_3(R)$ has the intersection property of quasi-ideals.*

Proof. Let X be a finite nonempty subset of $SU_3(R)$. Then

$$XSU_3(R) \subseteq \left\{ \begin{bmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| \, x \in R \right\}$$

and

$$SU_3(R)X \subseteq \left\{ \begin{bmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| \, x \in R \right\}.$$

To show $SU_3(R)X \cap (\mathbb{Z}X + XSU_3(R)) \subseteq \mathbb{Z}X + (SU_3(R)X \cap XSU_3(R))$, let $M \in SU_3(R)X \cap (\mathbb{Z}X + XSU_3(R))$. So $M \in SU_3(R)X$ and $M =$

$Q + N$, where $Q \in \mathbb{Z}X$ and $N \in XSU_3(R)$. Therefore, $Q = M - N \in \left\{ \begin{bmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid x \in R \right\} \bigcap \mathbb{Z}X.$

Define $\varphi : \left(\left\{ \begin{bmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid x \in R \right\}, + \right) \to (R, +)$ by

$$\varphi\left(\begin{bmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) = x \text{ for all } x \in R.$$

Then φ is a group isomorphism. Since φ is isomorphism and $XSU_3(R)$, $SU_3(R)X$ and $\left\{ \begin{bmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid x \in R \right\} \bigcap \mathbb{Z}X$ are subgroups of $\left\{ \begin{bmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid x \in R \right\}$, we have that $\varphi(XSU_3(R))$, $\varphi(SU_3(R)X)$ and $\varphi\left(\left\{ \begin{bmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid x \in R \right\} \bigcap \mathbb{Z}X \right)$ are subgroups of R. Therefore, by assumption, there exist $a, b, c \in R$ such that

$$\varphi(XSU_3(R)) = aR, \quad \varphi(SU_3(R)X) = bR$$

and

$$\varphi\left(\left\{ \begin{bmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid x \in \mathbb{Z} \right\} \bigcap \mathbb{Z}X \right) = cR.$$

Since $M = Q + N$,

$$\varphi(M) = \varphi(Q) + \varphi(N) \in bR \bigcap (aR + cR) \subseteq cR + (aR \bigcap bR)$$

by Lemma 2.1(iv). Therefore, $\varphi(M) = y + z$ for some $y \in bR$ and $z \in aR \cap cR$. Since φ is isomorphism, there exist $A \in \left\{ \begin{bmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| \, x \in \mathbb{Z} \right\}$ $\cap \, \mathbb{Z}X$ and $B \in XSU_3(R) \cap SU_3(R)X$ such that $y = \varphi(A)$ and $z = \varphi(B)$. Since φ is one-to-one, this implies that $M = A + B \in \mathbb{Z}X + XSU_3(R) \cap SU_3(R)X$. Therefore, $SU_3(R)X \cap (\mathbb{Z}X + XSU_3(R)) \subseteq \mathbb{Z}X + (SU_3(R)X \cap XSU_3(R))$. By Proposition 1.3, $SU_n(R)$ has the intersection property of quasi-ideals. □

Lemma 2.3. *Let R be an integral domain. If $n \geq 4$, then the ring $SU_n(R)$ does not have the intersection property of quasi-ideals.*

Proof. Let 1 be an identity of R and $A, B \in SU_n(R)$ be defined by

$$A = \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 1 \\ 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 1 \\ 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus,

$$SU_n(R)\{A, B\} = \left\{ \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 & x \\ 0 & \cdots & 0 & 0 & 0 & y \\ 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix} \middle| \, x, y \in R \right\},$$

$$\{A, B\}SU_n(R) = \left\{ \begin{bmatrix} 0 & \cdots & 0 & 0 & x & y \\ 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix} \middle| x, y \in R \right\}$$

and

$$\mathbb{Z}\{A, B\} = \left\{ \begin{bmatrix} 0 & \cdots & 0 & k & l & 0 \\ 0 & \cdots & 0 & 0 & 0 & l \\ 0 & \cdots & 0 & 0 & 0 & k \\ 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix} \middle| k, l \in \mathbb{Z} \right\}.$$

Then

$$\mathbb{Z}\{A, B\} + (SU_n(R)\{A, B\} \cap \{A, B\}SU_n(R))$$

$$= \left\{ \begin{bmatrix} 0 & \cdots & 0 & k & l & x \\ 0 & \cdots & 0 & 0 & 0 & l \\ 0 & \cdots & 0 & 0 & 0 & k \\ 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix} \middle| k, l \in \mathbb{Z}, x \in R \right\}. \qquad (*)$$

Let $C, D \in SU_n(R)$ be defined by

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 1 & 1 \\ 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then

$$CA = -B + AD = \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 & 1 \\ 0 & \cdots & 0 & 0 & 0 & -1 \\ 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}.$$

By (*), we have

$$CA = -B + AD \notin \mathbb{Z}\{A, B\} + (SU_n(R)\{A, B\} \cap \{A, B\}SU_n(R)).$$

But $CA = -B + AD \in SU_n(R)\{A, B\} \cap (\mathbb{Z}\{A, B\} + \{A, B\}SU_n(R)),$ so

$$SU_n(R)\{A, B\} \cap (\mathbb{Z}\{A, B\} + \{A, B\}SU_n(R))$$

$$\not\subseteq \mathbb{Z}\{A, B\} + (SU_n(R)\{A, B\} \cap \{A, B\}SU_n(R)).$$

By Proposition 1.3, $SU_n(R)$ does not have the intersection property of quasi-ideals. □

Now we are ready to prove our main result.

Theorem 2.4. *Let R be a Euclidean domain such that every additive subgroup is an ideal of R. Then the ring $SU_n(R)$ has the intersection property of quasi-ideals if and only if $n \leq 3$.*

Proof. If $n \leq 2,$ then $SU_n(R)$ is a zero ring. It implies that $SU_n(R)$ has the intersection property of quasi-ideals.

If $n = 3,$ then by Lemma 2.2, we have that $SU_3(R)$ has the intersection property of quasi-ideals.

For the converse, assume that $n \geq 4,$ by Lemma 2.3, the $SU_n(R)$ does not have the intersection property of quasi-ideals.

Hence, the theorem is completely proved. □

Example 2.5. The ring $\mathbb{Z}$ is the example of ring such that every additive subgroup is an ideal. By Theorem 2.4, we have that the ring $SU_n(\mathbb{Z})$ has the intersection property of quasi-ideals if and only if $n \leq 3$.

Remark 2.6. The authors in [2] show that every principal quasi-ideal of the ring $SU_n(R)$ has the intersection property. This implies that quasi-ideal in $SU_n(R)$ which does not have the intersection property must be generated by at least two elements. In the proof of Lemma 2.3, we see that there exist quasi-ideals in $SU_n(R)$ $(\text{case } n \geq 4)$ which does not have the intersection property generated by two elements.

Acknowledgment

This paper was supported by Algebra and Applications Research Unit, Prince of Songkla University.

References

[1] R. Chinram and Y. Kemprasit, The intersection property of quasi-ideals in generalized rings of strictly upper triangular matrices, East-West J. Math. 3 (2001), 201-210.

[2] R. Chinram and C. Rungsaripipat, The intersection property of quasi-ideals in rings of strictly upper triangular matrices over $m\mathbb{Z}$, Adv. Algebra Anal. 1 (2006), 123-131.

[3] Y. Kemprasit and P. Juntarakhajorn, Quasi-ideals of rings of strictly upper triangular matrices over a field, Ital. J. Pure Appl. Math. 12 (2002), 97-104.

[4] Y. Kemprasit and R. Chinram, Generalized rings of linear transformations having the intersection property of quasi-ideals, Vietnam J. Math. 30 (2002), 283-288.

[5] O. Steinfeld, On ideal-quotients and prime ideals, Acta Math. Acad. Sci. Hung. 4 (1953), 289-298.

[6] O. Steinfeld, Quasi-ideals in Rings and Semigroups, Akadémiai Kiadó, Budapest, 1978.

[7] H. J. Weinert, On quasi-ideals in rings, Acta Math. Hungar. 43 (1984), 85-99.

[8] M. C. Zhang, Y. Q. Chen and Y. H. Li, The intersection property of quasi-ideals in rings and semigroup rings, SEA Bull. Math. 20 (1996), 117-122.