Far East Journal of Dynamical Systems

Published Online: August 2016
http://dx.doi.org/10.17654/DS028020077
Volume 28, Number 2, 2016, Pages 77-88 **ISSN: 0972-1118**

RELATIONSHIP BETWEEN ω-LIMIT SETS AND MINIMAL SETS

Liying Wang, Likun Kang and Long Wang[*]

Department of Mathematics
Zhangjiakou Vocational and Technical College
Zhangjiakou, Hebei, P. R. China
e-mail: wly031230@163.com

[*]Department of Mathematics
Kennesaw State University
Marietta, GA 30060, U. S. A.
e-mail: lwang17@kennesaw.edu

Abstract

In this paper, we study the relation between ω-limit sets and minimal sets. It is known that every minimal set is an ω-limit set. However, not every ω-limit set is a minimal set, although it contains a minimal set. We would like to know under what condition, an ω-limit set turns out to be a minimal set. We prove a necessary and sufficient condition under which every ω-limit set is minimal. We also establish a sufficient condition for an ω-limit set to be minimal.

1. Introduction

Throughout this paper, I denotes a compact interval and $C(I, I)$ denotes the collection of all continuous functions f which map I to itself. For

Received: September 13, 2013; Revised: April 14, 2015; Accepted: April 16, 2015
2010 Mathematics Subject Classification: Primary 58F08.
Keywords and phrases: periodic points, recurrent points, ω-limit set, minimal set.
Communicated by K. K. Azad

$f \in C(I, I)$, the nth iterate of f, denoted by f^n, is defined inductively by setting $f^0(x) = x$ and $f^n(x) = f(f^{n-1}(x))$ for $n \geq 1$. For $x \in I$, $\omega(x, f)$ denotes the set of all limit points of the sequence $\{f^n(x)\}_{n \geq 0}$ and $\Lambda(f) = \bigcup_{x \in I} \omega(x, f)$. Let $R(f)$ denote the set of all recurrent points of f and $AP(f)$ denote the set of all almost periodic points of f.

The main aim of this paper is to study the relationship between ω-limit sets and minimal sets. It is well-known that every minimal set is an ω-limit set. However, not every ω-limit set is a minimal set, although it contains a minimal set. We prove that $R(f) = \Lambda(f)$ is a necessary and sufficient condition under which every ω-limit set is minimal.

Another important result in this paper is a sufficient condition for an ω-limit set to be a minimal set. A special case of this result is that if an ω-limit set consists of only periodic points, then it is a periodic orbit (a finite minimal set).

2. Preliminaries and Known Results

Recall that $f \in C(I, I)$ is a continuous function of the compact interval I into itself and f^n is the nth iterate of f. In this section, we state the definitions and notation and list some important theorems which will be used later either explicitly or implicitly. Further, although we restrict our attention to functions of a compact interval, many of the ideas are applicable to mappings of a compact metric space and some of the results are also valid for mappings of a compact metric space.

Definition 2.1. For a given point $x \in I$, the trajectory or orbit of f at x, denoted by $O(x, f)$, is the sequence $\{f^n(x)\}_{n \geq 0}$. Sometimes we view $O(x, f)$ as a sequence and sometimes as a set. However, the context will always indicate which is meant.

In the case where $O(x, f)$ has a finite range, x is called an *eventually periodic point* of f. The set of all eventually periodic points is denoted by $EP(f)$.

Definition 2.2. A point $x \in I$ is called a *fixed point* of f if $f(x) = x$. The set of all fixed points of f is denoted by $F(f)$. If m is a positive integer, a point $x \in I$ is called a *periodic point* of f of period m, if $f^m(x) = x$ and $f^i(x) \neq x$ for $1 \leq i \leq m - 1$. The set of all periodic points of f is denoted by $P(f)$.

Definition 2.3. A point $x \in I$ is called an *almost periodic point* of f if for every open set U containing x, there exists a positive integer N such that if $f^m(x) \in U$ and $m \geq 0$, then $f^{m+k}(x) \in U$ for some k, $0 \leq k \leq N$. The set of all the almost periodic points of f is denoted by $AP(f)$.

Definition 2.4. A point $x \in I$ is called a *recurrent point* of f if for every open set U containing x, there exists a positive integer n such that $f^n(x) \in U$. The set of all recurrent points of f is denoted by $R(f)$.

Definition 2.5. For a given point $x \in I$, the ω-limit set of x, denoted by $\omega(x, f)$, is the set of all points y for which there exists a sequence $\{f^{n_k}(x)\}$ such that as $k \to \infty$, $n_k \to \infty$ and $f^{n_k}(x) \to y$. A point $y \in \omega(x, f)$ is called an ω-*limit point*. The set of all ω-limit points is denoted by $\Lambda(f)$, $\Lambda(f) = \bigcup_{x \in I} \omega(x, f)$.

Definition 2.6. A subset M of I is said to be a *minimal set* with respect to f if it is nonempty, closed and invariant and if no proper subset has these three properties.

Definition 2.7. f is said to be *turbulent* if there exist closed subintervals J and K such that $J \cap K$ contains at most one point, and $J \cup K \subseteq f(J) \cap f(K)$. If $J \cap K = \varnothing$, then f is said to be *strictly turbulent*.

Definition 2.8. f is said to be *chaotic* if f^n is turbulent for some positive integer n. If f is not chaotic, then f is said to be *non-chaotic*.

Definition 2.9. The symbol space is the metric space (Σ_2, d), where $\Sigma_2 = \{0, 1\}^N$, N is the set of non-negative integers and

$$d(s, t) = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i}$$

for $s = (s_0 s_1 s_2 \cdots)$, $t = (t_0 t_1 t_2 \cdots) \in \Sigma_2$.

The one-sided shift map $\sigma : \Sigma_2 \mapsto \Sigma_2$ is defined by $\sigma(s_0 s_1 s_2 \cdots) = (s_1 s_2 \cdots)$.

Note that σ is continuous. Now we are in the position to list some known results to be used later. They can be found in [3] unless other references are specified.

Theorem 2.1. *The following statements are equivalent*:

(1) *f is chaotic*,

(2) *f has a periodic point of period which is not a power of* 2,

(3) *there exists* $x \in I$ *such that* $\omega(x, f)$ *properly contains a periodic orbit*,

(4) *there exists* $x \in I$ *such that* $\omega(x, f)$ *is countably infinite*,

(5) $AP(f) \neq R(f)$,

(6) *there exists a compact set* $x \subseteq I$ *such that* $f^m(X) = X$ *for some positive integer m and a continuous map h of X onto* Σ_2 *such that each point of* Σ_2 *is the image of at most two points of X and*

$$h \circ f^m(x) = \sigma \circ h(x)$$

for all $x \in X$.

Theorem 2.2. *If f is chaotic, then there exists* $x \in I$ *such that* $\omega(x, f)$ *contains infinitely many minimal sets.*

3. Relations between ω-limit Sets and Minimal Sets

In this section, we establish a necessary and sufficient condition that ω-limit set be minimal for every $x \in I$. Then we give some sufficient conditions for an ω-limit set to be minimal. We start with the following simple theorem which can be found in [4].

Theorem 3.1. $\omega(x, f)$ *is a finite minimal set for every* $x \in I$ *if and only if* $P(f) = \Lambda(f)$.

Proof. Assume $\omega(x, f)$ is a finite minimal set for every $x \in I$. Then it is easy to see that $\omega(x, f)$ is a periodic orbit. So $\omega(x, f) \subseteq P(f)$ for each $x \in I$. Thus, we have

$$\Lambda(f) = \bigcup_{x \in I} \omega(x, f) \subseteq P(f) \subseteq \Lambda(f).$$

So $P(f) = \Lambda(f)$.

Conversely, if $P(f) = \Lambda(f)$, due to the fact, $P(f) \subseteq AP(f) \subseteq R(f) \subseteq \Lambda(f)$, we have $AP(f) = R(f) = P(f)$ which implies that *f* is non-chaotic by Theorem 2.1.

Now let $x \in I$. Then $\omega(x, f) \subseteq P(f)$, since $\Lambda(f) = P(f)$. So $\omega(x, f)$ contains a periodic orbit *M*. By Theorem 2.1, it cannot contain *M* properly. Therefore, $\omega(x, f) = M$ which is a finite minimal set. □

In order to prove a necessary and sufficient condition that every ω-limit set is minimal for a given *f*, we need the following lemma:

Lemma 3.2. *If* $R(f)$ *is closed, then f is non-chaotic.*

Proof. Suppose to the contrary that *f* is chaotic. Then, by Theorem 2.1(6), there exists a compact subset *X* of *I* such that $f^m(X) = X$ for some positive integer m and there is a continuous map h of X onto Σ_2 such that each point of Σ_2 is the image of at most two points of X and

$$h \circ f^m(x) = \sigma \circ h(x) \tag{3.1}$$

for all $x \in X$. Since $R(f^m) = R(f)$ is closed, $X \cap R(f^m)$ is closed.

Claim. $R(\sigma) = h(X \cap R(f^m))$.

Proof of Claim. If $x \in X \cap R(f^m)$, then there exists a sequence $\{f^{n_k m}(x)\}$ such that as $k \to \infty$, $n_k \to \infty$ and $f^{n_k m}(x) \to x$. It follows from (3.1) that $\sigma^{n_k}(h(x)) = h(f^{n_k m}(x))$ which converges to $h(x)$. This shows that $h(x) \in R(\sigma)$. On the other hand, if $y \in R(\sigma)$ and y has a unique x such that $h(x) = y$, then $\sigma^{n_k}(y) \to y$ for some $n_k \to \infty$. Again, it follows from (3.1) that $h(f^{n_k m}(x)) = \sigma^{n_k}(y)$. Without loss of generality, assume $f^{n_k m}(x) \to x_1$. Then $h(x_1) = y$ implies that $x_1 = x$. So $x \in X \cap R(f^m)$. Now suppose that there are two points $x_1, x_2 \in X$ with $h(x_1) = h(x_2) = y$. Then we have $h(f^{n_k m}(x_1)) = \sigma^{n_k}(y)$ and $h(f^{n_k m}(x_2)) = \sigma^{n_k}(y)$. Again, without loss of generality, assume $f^{n_k m}(x_1)$ and $f^{n_k m}(x_2)$ are two convergent sequences which converge to either x_1 or x_2. Assume $x_1 \notin R(f^m)$. Then $f^{n_k m}(x_1) \to x_2$ and $x_2 \in \omega(x_1, f^m)$. Hence, $f^{n_k m}(x_2) \to x_2$ since otherwise if $f^{n_k m}(x_2) \to x_1$, then $x_1 \in \omega(x_2, f^m) \subseteq \omega(x_1, f^m)$ would imply $x_1 \in R(f^m)$. So $x_2 \in R(f^m)$. This shows that $R(\sigma) = h(X \cap R(f^m))$. It follows from Claim that $R(\sigma)$ is closed. Thus, $R(\sigma) = \overline{R(\sigma)}$. But, in Σ_2, we know $\overline{R(\sigma)} = \overline{P(\sigma)} = \Sigma_2$. So $R(\sigma) = \Sigma_2$. This is a contradiction, because obviously, $(1, 0, 0, 0, \ldots) \notin R(\sigma)$. This proves that f is non-chaotic. □

Now we are ready to prove the following theorem:

Theorem 3.3. $\omega(x, f)$ *is minimal for each* $x \in I$ *if and only if* $R(f) = \Lambda(f)$.

Proof. Assume that $\omega(x, f)$ is minimal for each $x \in I$. Then, for every $y \in \omega(x, f)$, $\overline{O(y, f)} = \omega(x, f)$ which implies that $y \in AP(f)$. So

$$\omega(x, f) \subseteq AP(f)$$

and

$$\Lambda(f) = \bigcup_{x \in I} \omega(x, f) \subseteq AP(f) \subseteq R(f).$$

It is trivial that $R(f) \subseteq \Lambda(f)$. Therefore,

$$R(f) = \Lambda(f).$$

Conversely, if $R(f) = \Lambda(f)$, then $R(f)$ is a closed set which implies that f is non-chaotic by Lemma 3.2. Let $x \in I$. If $\omega(x, f)$ is finite, then $\omega(x, f)$ is a periodic orbit and therefore a minimal set. Assume $\omega(x, f)$ is infinite. It also follows from the assumption $\Lambda(f) = R(f)$ that $\Lambda(f) = AP(f)$. Hence, $\omega(y, f)$ is minimal for all $y \in \omega(x, f)$. Since f is non-chaotic, $\omega(y, f)$ contains a unique minimal set. Thus, $\omega(x, f) = \bigcup_{y \in \omega(x, f)} \omega(y, f) = \omega(y, f)$ for any $y \in \omega(x, f)$. This shows that $\omega(x, f)$ is minimal. □

Now we give some sufficient conditions for an ω-limit set to be minimal. First, we need the following lemma:

Lemma 3.4. *Let* $x \in I$. *If* $\omega(x, f) \subseteq AP(f)$, *then* $\omega(x, f)$ *is either a finite set or a Cantor set.*

Proof. Suppose that $\omega(x, f)$ is infinite. First, we show all points of $\omega(x, f)$ are limit points of $\omega(x, f)$. Let $y \in \omega(x, f)$. If y is periodic, then it is a limit point of $\omega(x, f)$. If y is not a periodic point, since $y \in AP(f)$, $\omega(y, f)$ is an infinite minimal set which is a subset of $\omega(x, f)$. Since $y \in \omega(y, f)$ is a limit point of $\omega(y, f)$, it is also a limit point of $\omega(x, f)$.

It remains to show that $\omega(x, f)$ is nowhere dense. Suppose to the contrary that it contains an interval (a, b). Then $f^N(x) \in (a, b)$ for some positive integer N. Since $\omega(x, f) \subseteq AP(f)$, $f^N(x) \in AP(f)$ and therefore $\omega(f^N(x), f)$ is a minimal set. Thus, $\omega(x, f) = \omega(f^N(x), f)$ is a minimal set. This contradicts that a minimal set is a Cantor set. So $\omega(x, f)$ is either a finite set or a Cantor set. □

Theorem 3.5. *Let* $x \in I$. *If* $\omega(x, f) \subseteq P(f)$, *then* $\omega(x, f)$ *is finite and a periodic orbit.*

Proof. For simplicity, assume $I = [0, 1]$. Suppose that $\omega(x, f)$ is infinite. By Lemma 3.4, it is a Cantor set. Now there are two possibilities:

(i) The periods of points in $\omega(x, f)$ are bounded.

(ii) The periods of points in $\omega(x, f)$ are unbounded.

Let us assume case (i) first. Let K be the least upper bound of the periods and $N = K!$. Then we have

$$\omega(x, f) = \bigcup_{j=0}^{N-1} \omega(f^j(x), f^N)$$

and

$$f(\omega(f^j(x), f^N)) = \omega(f^{j+1}(x), f^N).$$

Therefore, each $\omega(f^j(x), f^N)$ is infinite and therefore a Cantor set for $1 \le j \le N-1$. Choose an interval (a, b) such that $(a, b) \cap \omega(x, f^N) = \varnothing$, $\omega(x, f^N) \cap [0, a) \neq \varnothing$ and $\omega(x, f^N) \cap [b, 1) \neq \varnothing$ and choose a point c

in (a, b). Let $A = \omega(x, f^N) \cap [0, c]$, $B = \omega(x, f^N) \cap [c, 1]$. Then clearly A and B are closed, disjoint and $\omega(x, f^N) = A \cup B$. Moreover, $f^N(A) = A$ and $f^N(B) = B$. This is impossible.

Now assume the second case. Let $P_n = \{y, f^n(y) = y\}$ and $X_n = \omega(x, f) \cap P_n$. Then $\omega(x, f) = \bigcup_{n \geq 1} X_n$ and each X_n is closed. So, by the Baire Category Theorem, there exist a, b such that $\omega(x, f) \cap (a, b) = X_n \cap (a, b)$ which is not empty for some n. Since

$$\omega(x, f) = \bigcup_{j=0}^{n-1} \omega(f^j(x), f^n),$$

$$\omega(f^j(x), f^n) \bigcap (a, b) \neq \varnothing$$

for some j. Since

$$\omega(f^j(x), f^n) \bigcap (a, b) \subseteq \omega(x, f) \bigcap (a, b) = X_n \bigcap (a, b),$$

we have

$$\omega(f^j(x), f^n) \bigcap (a, b) \subseteq F(f^n).$$

Similarly, since $\omega(x, f)$ is a Cantor set, $\omega(f^j(x), f^n)$ is a Cantor set. Hence, we can find disjoint open intervals (c_1, d_1) and (c_2, d_2) in (a, b) such that

$$\omega(f^j(x), f^n) \bigcap (c_1, d_1) = \varnothing,$$

$$\omega(f^j(x), f^n) \bigcap (c_2, d_2) = \varnothing,$$

$$\omega(f^j(x), f^n) \bigcap (d_1, c_2) \neq \varnothing,$$

$$\omega(f^j(x), f^n) \bigcap (a, c_1) \neq \varnothing.$$

Choose c in (c_1, d_1) and d in (c_2, d_2). Then $c, d \notin \omega(f^j(x), f^n)$.

Let $A = \omega(f^j(x), f^n) \cap [c, d]$, $B = \omega(f^j(x), f^n) \cap ([0, c] \cup [d, 1])$. Then it follows from the choices of c and d that A, B are closed, disjoint and $\omega(f^j(x), f^n) = A \cup B$. Moreover, we claim that

$$f^n(A) \subseteq A, \quad f^n(B) \subseteq B.$$

In fact, if $y \in A$, then $y \in X_n$ and $f^n(y) = y$ which gives $f^n(A) = A$. Let $y \in B$. Assume that $f^n(y) \in A$. Then $f^{2n}(y) = f^n(y)$ and $f^{in}(y) = f^n(y)$ for all $i \geq 1$ which implies that $f^{in}(y) \neq y$ for all $i \geq 1$. But since $y \in P(f) = P(f^n)$, this is a contradiction. So $f^n(B) \subseteq B$.

Now we have that $\omega(f^j(x), f^n) = A \cup B$, where A, B are closed, disjoint and forward invariant under f^n. This is also impossible.

So $\omega(x, f)$ is a finite set and therefore it is a periodic orbit. □

To prove the next result in this section, we need to establish the following lemma:

Lemma 3.6. *Let $x \in I$. If $\omega(x, f)$ properly contains an infinite minimal set M and $f(\omega(x, f) \backslash M) \subseteq \omega(x, f) \backslash M$, then $M \subseteq \overline{\omega(x, f) \backslash M}$.*

Proof. It is clear that there exists $y \in M$ such that $y \in \overline{\omega(x, f) \backslash M}$. We proceed by contradiction. Suppose that there exists $z \in M$ such that $z \notin \overline{\omega(x, f) \backslash M}$. Then we can find an open neighborhood U of z for which

$$U \bigcap (\omega(x, f) \backslash M) = \varnothing. \tag{3.2}$$

Since M is minimal, there exists a positive integer N such that $f^N(y) \in U$. $y \in \overline{\omega(x, f) \backslash M}$ implies that there exists a sequence $\{y_k\} \subseteq \omega(x, f) \backslash M$ such that $y_k \to y$. Thus, $f^N(y_k) \to f^N(y)$. So $f^N(y_k) \in U$ for sufficiently large k. Since $y_k \in \omega(x, f) \backslash M$ which is invariant under f,

$f^N(y_k) \in \omega(x, f)\backslash M$. It follows that

$$U \bigcap \omega(x, f)\backslash M \neq \varnothing$$

which contradicts (3.2). Hence, $M \subseteq \overline{\omega(x, f)\backslash M}$. □

Theorem 3.7. *Let* $x \in I$. *If* $\omega(x, f) \subseteq AP(f)$ *and there exists an interval* (a, b) *such that* $\omega(x, f) \cap (a, b) \neq \varnothing$ *and* $\omega(x, f) \cap (a, b) \subseteq \omega(y, f) \cap (a, b)$ *for some* $y \in \omega(x, f)$, *then* $\omega(x, f)$ *is a minimal set.*

Proof. The proof is a consequence of Lemma 3.6. Since under the hypothesis, if $\omega(x, f)\backslash\omega(y, f) \neq \varnothing$, then some points of $\omega(y, f)$ inside (a, b) would be isolated from $\omega(x, f)\backslash\omega(y, f)$. So $\omega(x, f) = \omega(y, f)$ which is minimal. □

Acknowledgement

The authors thank the anonymous referees for their valuable suggestions which led to the improvement of the manuscript.

References

[1] S. J. Agronsky, A. M. Bruckner, J. G. Ceder and T. L. Pearson, The structure of ω-limit sets for continuous functions, Real Anal. Exchange 15 (1989/90), 483-510.

[2] L. S. Block, Continuous maps of the interval with finite nonwandering set, Trans. Amer. Math. Soc. 240 (1978), 221-230.

[3] L. S. Block and W. A. Coppel, One-dimensional Dynamics, Spring-Verlag, 1991.

[4] L. S. Block and E. M. Coven, ω-limit sets for maps of the interval, Dynamical Systems and Ergodic Theory 156 (1986), 513-515.

[5] G. D. Birkhoff, Dynamical Systems, Amer. Math. Soc. Colloquium Publication, 1966.

[6] A. M. Bruckner and J. Ceder, Chaos in terms of the map $x \mapsto \omega(x, f)$, Pacific J. Math. 10 (1992), 63-96.

[7] A. M. Bruckner and T. Hu, Equicontinuity of iterates of an interval map, Tamkang J. Math. 21 (1990), 287-294.

[8] A. M. Bruckner and J. Smital, The structure of ω-limit sets for continuous maps of the interval, Math. Bohem. 117 (1992), 42-47.

[9] P. Collet and J. P. Eckman, Iterated maps on the interval as dynamical systems, Progress in Physics, 1, Birkhäuser, Basel, 1980.

[10] E. M. Coven and G. A. Hedlund, $\overline{P} = \overline{R}$ for maps of the interval, Proc. Amer. Math. Soc. 79 (1980), 316-318.

[11] R. L. Devaney, Introduction to Chaotic Dynamical Systems, Benjamin-Cummings, Menlo Park, 1986.

[12] M. J. Evan, P. D. Humke, C. M. Lee and R. J. O'Malley, Characterization of turbulent one-dimensional mappings via ω-limit sets, Trans. Amer. Math. Soc. 326 (1991), 261-280.

[13] S. H. Li, ω-chaos and topological entropy, Trans. Amer. Math. Soc. 339 (1993), 243-249.

[14] S. H. Li, Some properties of continuous type 2^{∞} unimodal mapping on $[0, 1]$, J. China Univ. Sci. Tech. 16 (1986), 8-19.

[15] T. Y. Li and J. A. Yorke, ω-limit sets for maps of the interval, Amer. Math. Monthly 82 (1975), 985-992.

[16] A. N. Šarkovskii, Attracting and attracted sets, Dokl. Akad. Nauk SSSR 160 (1965), 1036-1038.

[17] A. N. Šarkovskii, The behaviour of a mapping in the neighbourhood of an attracting set, Ukrain. Mat. Zh. 18 (1966), 60-83.

[18] A. N. Sarkovskii, Continuous mapping on the set of limit points of an iteration sequence, Ukrain. Mat. Zh. 18 (1966), 127-130.

[19] J. Smital, Chaotic functions with zero topological entropy, Trans. Amer. Math. Soc. 297 (1986), 269-282.

[20] L. Wang, A criterion for finitely many points to belong to a common ω-limit set, Far East J. Dyn. Syst. 21(2) (2013), 69-82.

[21] J. C. Xiong, Sets of recurrent points of continuous maps of the interval, Proc. Amer. Math. Soc. 95 (1985), 491-494.

[22] J. C. Xiong, The attracting center of a continuous self-map of the interval, Ergodic Theory Dynam. Systems 8 (1988), 205-213.