Advances and Applications in Fluid Mechanics
© 2016 Pushpa Publishing House, Allahabad, India
Published Online: August 2016
http://dx.doi.org/10.17654/FM019030619
Volume 19, Number 3, 2016, Pages 619-630 **ISSN: 0973-4686**

APPLICATION OF THE MODEL OF CANUTO AND DUBOVIKOV TO THE DECREASING OF A TWO-DIMENSIONAL HOMOGENEOUS TURBULENCE

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Abstract

The dynamic model of Canuto and Dubovikov was designed to describe fully developed turbulence which also corresponds to our study.

A family of models was developed to describe the nonlinear interactions in spectral space for an incompressible homogeneous turbulence (DIA, TFM, EDQNM, ...). More recently, the dynamic model of Canuto and Dubovikov has also been developed in the context of spectral representation of nonlinear mechanisms.

In its design, the dynamic model of Canuto and Dubovikov is based on Wyld equations [1] which are exact formal solutions of the Navier-

Received: March 19, 2016; Accepted: June 24, 2016
2010 Mathematics Subject Classification: 76F05, 76F06, 76F35, 76F55, 76M22.
Keywords and phrases: homogeneous turbulence, dynamic model, two-dimensional turbulence, enstrophy, palinstrophy.
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Communicated by Shahrdad G. Sajjadi

Stokes equations. These equations lead to represent the nonlinear interactions through two mechanisms leading force f^t in charge of a turbulent flow of energy $\pi(k)$ and a dependent viscosity $\nu_d(k)$ in spectral space wave vector $\vec{k}$. Through the theory of renormalization group (RNG), π and ν_d are expressed using infinite series π. Finally, the series giving π contain like (IR) divergences that are not renormalizable, and to express π, the dynamic model of Canuto and Dubovikov uses the concept of energy locality [1-5].

Introduction

The application of the dynamic model of Canuto and Dubovikov to two-dimensional homogeneous turbulence is considered again directed by a relatively original contribution towards the study of the asymptotic behavior (with large *T*) of such a turbulence. Through these results of digital simulation of Chasnov [6], a framework is offered to evaluate the results of the dynamic model relating to the asymptotic behavior of this turbulence.

Initialization of calculations and physical parameters: In all the considered calculations, the following initial spectrum of energy is used:

$$E(k,\ 0) = \frac{1}{2} a_s u_0^2 k_p^{-1} \left(\frac{k}{k_p}\right)^{2s+1} \exp\left[-\left(s + \frac{1}{2}\right)\left(\frac{k}{k_p}\right)^2\right], \tag{1}$$

where the constant of standardization a_s is given by:

$$a_s = (2s+1)^{s+1}/2^s s!. \tag{2}$$

This choice allows us to define $\frac{u_0^2}{2}$ as being the initial kinetic energy:

$$E_{kin}(0) = \frac{u_0^2}{2} = \int_0^{+\infty} E(k,\ 0)dk \tag{3}$$

which is obviously the number of waves for which $E(k,\ 0)$ is maximum.

The dynamic Reynolds number is:

$$R(t) = \frac{u(t)l(t)}{\nu}, \tag{4}$$

where $u(t)$ is the square root of the variance speed:

$$u(t) = \langle u^2 \rangle^{1/2}(t) \tag{5}$$

and $l(t)$ is a dynamic scale length that is equal to the relationship between speed $u(t)$ and the vorticity $\omega(t)$, defined as square root of the variance $\langle \omega^2 \rangle$, that is:

$$l(t) = u(t)/\omega(t), \tag{6}$$

$$\omega(t) = l\langle \omega^2 \rangle^{1/2}. \tag{7}$$

A simple calculation makes it possible to express the initial values $u(0)$, $\omega(0)$, $l(0)$ and $R(0)$ according to the initial spectrum:

$$u = (0) = u_0, \quad \omega(0) \equiv \omega_0 = \sqrt{\frac{2s+2}{2s+1}}\, u_0 k_p, \tag{8}$$

$$l(0) \equiv l_0 = \sqrt{\frac{2s+1}{2a+2}}\, k_p^{-1}, \tag{9}$$

$$R(0) = \sqrt{\frac{2s+1}{2a+2}}\, \frac{u_0}{k_p \nu}. \tag{10}$$

Evolution of a two-dimensional turbulence with respect to small Reynolds number: In this paragraph, the evolution of turbulence is discussed with respect to small initial Reynolds number. For large Reynolds numbers, Batchelor had announced that the kinetic energy becomes constant with large *T* and that the enstrophy behaves according to time *T* in T^{-2}.

The analysis of the decrease of a two-dimensional turbulence can be done in a way similar to that of a 3D turbulence

$$E(k, t) \sim \pi B_2(t) k^3, \quad k \to 0, \tag{11}$$

where k is small $(k \to 0)$.

A dimensional analysis makes it possible to find laws of decrease for the kinetic energy and the enstrophy in the form [6]:

$$\langle u^2 \rangle \propto B_2(\nu t)^{-2}, \tag{12}$$

$$\langle \omega^2 \rangle \propto B_2(\nu t)^{-3}. \tag{13}$$

We recall that these behaviors are obtained by supposing that the variances $\langle u^2 \rangle$ and $\langle \omega^2 \rangle$ evolve linearly according to B_2 and depend only on kinematic viscosity ν and time T.

Materials and Methods

We solve numerically the equation:

$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right) E(k, t) = T(k, t)$$

governing the evolution of the energy spectrum $E(k)$. We combine the equations of initial energy spectrum in a classical form:

$$E(k, 0) = Ak^n \exp(-\alpha k^2),$$

where A, α and n are constants to be chosen according to the cases studied. Equations are solved numerically using a code written in FORTRAN. The temporal discretization is based on the Runge-Kutta fourth order, derivatives and integrals with respect to log k are evaluated, respectively, by using a finite difference scheme and using Simpson's method.

The wavenumber k is chosen according to the following expression known by octave:

$$k_i = k_{\min} 2^{i/f}, \quad i \in [nk_{\min}, nk_{\max}].$$

The resolution includes 220k modes and corresponds to $nk \min = 1$ and $nk \max = 220$. The parameter *f* is taken equal to 16.

Results

In this part, we deal with the application of the dynamic model in order to find usage of the digital simulation of equations of this model, the laws of decrease for energy and the enstrophy.

In the presence of the initial spectrum (1); the constant *s*, appearing in the expression of the initial spectrum, is equal to 3. The number of wave k_p is equal to 300 and the initial kinetic energy $\frac{u_0^2}{2}$ is equal to $\frac{1}{2}$.

In Figure 1, the evolution of the Reynolds number R is presented according to the time τ defined by:

$$\tau = \int_0^t dt(\omega^2)^{1/2}.$$

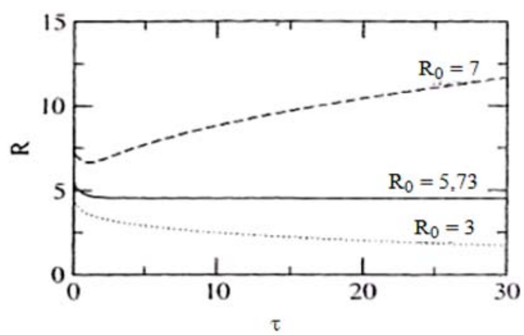

Figure 1. Evolution of the Reynolds number according to the time.

This time τ can be regarded as a measurement of the time of reversal of turbulence.

We show, through Figure 1, that there is a critical initial Reynolds number R_C for which the $R(t)$ number decreases according to time when $R_0 < R_C$ and increases when $R_0 > R_C$. For $R_0 = R_C$, $R(t)$ evolves as soon as T is higher than 5. τ is the value of R_C, our simulation makes it possible to find $R_C = 5{,}73$. A larger value $(R_C = 15{,}73)$ is found by Chasnov. But the evolution of $R(t)$ is qualitatively in agreement with that of Chasnov [6].

In Figure 2, the evolution of the laws of decrease of energy and the enstrophy are shown, $n(t)$ and $m(t)$ are exhibitors relating to these laws:

$$\langle u^2 \rangle \approx t^{n(t)}, \tag{14}$$

$$\langle \omega^2 \rangle \approx t^{m(t)}. \tag{15}$$

Then $n(t)$ and $m(t)$ are expressed as:

$$n(t) = \frac{d \ln \langle u^2 \rangle}{d \ln t}, \tag{16}$$

$$m(t) = \frac{d \ln \langle \omega^2 \rangle}{d \ln t}. \tag{17}$$

This allows us to write:

$$n(t) = -2\nu t \frac{\langle w^2 \rangle}{\langle u^2 \rangle}, \tag{18}$$

$$m(t) = -2\nu t \frac{\langle (\nabla\omega)^2 \rangle}{\langle \omega^2 \rangle}. \tag{19}$$

For $R_0 = R_C$, we assume that $R(t)$ tends to a constant value R'_C. Chasnov found analytical solutions for the evolution $\langle u^2 \rangle$ and $\langle \omega^2 \rangle$ that may be written as follows:

$$\langle u^2 \rangle = \frac{1}{2} \nu R_C'^2 t^{-1}, \tag{20}$$

$$\langle \omega^2 \rangle = \frac{1}{4} R_C'^2 t^{-2}. \tag{21}$$

Let us recall that the variances $\langle u^2 \rangle$ and $\langle (\nabla\omega)^2 \rangle$ are calculated according to the spectrum of energy $E(k)$.

For $R_0 = R_C$, Figure 2 shows that $n(t)$ tends towards an equal value with –1 and $m(t)$ tends towards a value equals to –2.

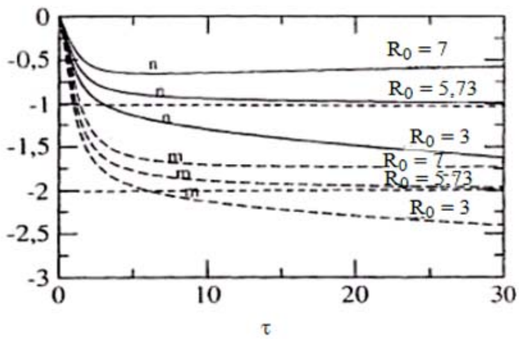

Figure 2. Evolution of the exhibitors for laws of decrease of energy and enstrophy.

For $R_0 = 7$, $n(t) > -1$ and $m(t) > -2$. For $R_0 = 3$, $n(t)$ tends towards values smaller than –1 while $m(t)$ tends towards values smaller than –2.

For $R_0 = R_C$, we suppose that $R(t)$ tends to a constant value R'_C.

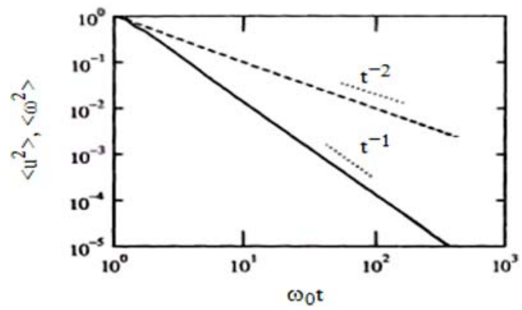

Figure 3. Evolution of the kinetic energy and the enstrophy for $R_0 = 5{,}73$.

We show in Figure 3 that the evolution of energy and the enstrophy, for $R_0 = R_C = 5{,}73,$ is well in agreement with the analytical results (20) and (21).

Decrease of a two-dimensional turbulence with respect to a large Reynolds number: We solve the equation of the dynamic model in the case of a two-dimensional homogeneous turbulence evolved by a large initial Reynolds number $R(0)$.

In our simulation, $R(0)$ varies between 32 and 4096 following 2^I $(5 \leq I \leq 12)$ relations. For that, 8 tests are conducted. The number k_p varies each time, according to $R(0)$; it is calculated using (10):

$$k_p = \sqrt{\frac{2s+1}{2s+2}}\frac{u_0}{R(0)\nu}.$$

We take s equal to 3.

In Figures 4 and 5, we represent the evolution of the kinetic energy $\langle u^2 \rangle$ and the enstrophy $\langle \omega^2 \rangle$, respectively, standardized by u_0^2 and ω_0^2.

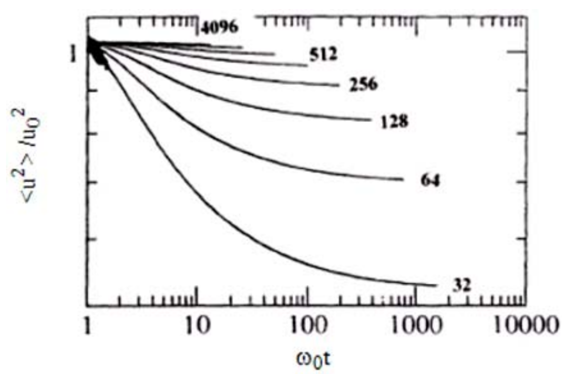

Figure 4. Evolution of the kinetic energy according to time for: R_0 = 32, 64, 128, 256, 512, 1024, 2048, 4096.

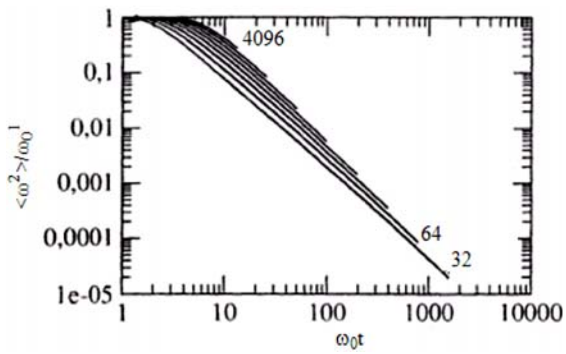

Figure 5. Evolution of the enstrophy according to time for: $R_0 = 32$, 64, 128, 256, 512, 1024, 2048, 4096.

The kinetic energy and the enstrophy decrease according to time in accordance with the basic equation governing the evolution of the kinetic energy and the enstrophy.

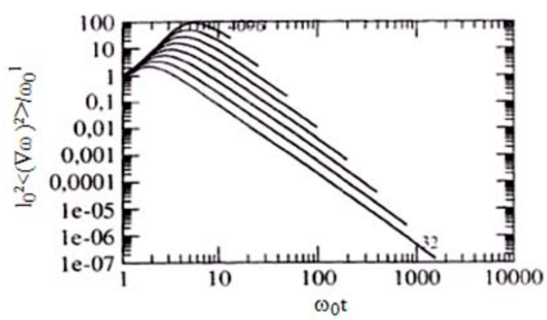

Figure 6. Evolution of the palinstrophy according to time for: $R_0 = 32$, 64, 128, 256, 512, 1024, 2048, 4096.

Let us notice that the palinstrophy (Figure 6) increases at the beginning of the evolution before decreasing when the time is large.

Figure 7 shows the evolution of exhibitor N relating to the decrease of the kinetic energy $(\langle u^2 \rangle \approx t^{n(t)})$ with great Reynolds number. There is no universal law for the asymptotic evolution of energy. In addition, the decrease of the kinetic energy becomes increasingly slow when $R(0)$ increases; this is explained by the increasingly large values of N.

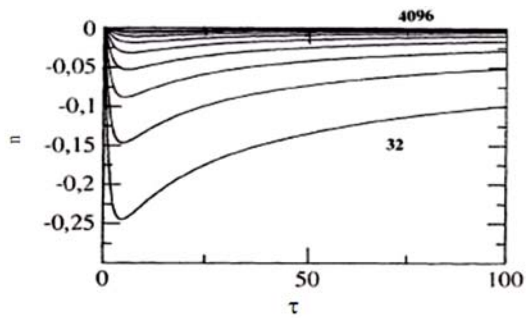

Figure 7. Evolution of the exhibitors n of the law of decrease of the kinetic energy for: $R_0 = 32, 64, 128, 256, 512, 1024, 2048, 4096$.

For $R(0) < 256$, exhibitor N decreases until a minimum grows at the end of the evolution.

For $R(0) > 256$, exhibitor N does not present any more minimum, it always increases.

The evolution of the exhibitor m according to time is presented in Figure 8. We notice that, for the whole of the tests, m tends towards –0.8.

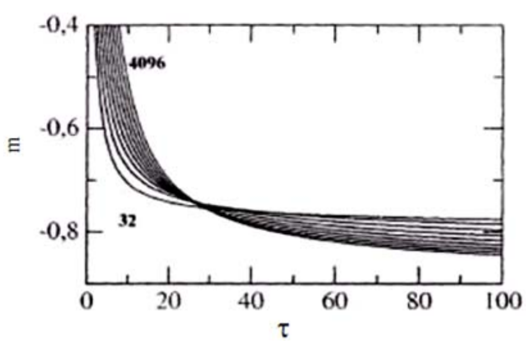

Figure 8. Evolution of the exhibitors m of the law of decrease of the kinetic energy for: $R_0 = 32, 64, 128, 256, 512, 1024, 2048, 4096$.

This value is blamed even by Chasnov, while indicating the behavior of the enstrophy in $T^{-0.37}$ suggested by Carnevale et al. [7] and the behaviors varying between $T^{-0.29}$ and $T^{-0.35}$ given by Dritschel [8].

The evolution of the $R(t)$ number is presented in Figure 9. We notice that this number increases according to time. This evolution conforms with the results because all the values of $R(0)$ are definitely higher than R_C.

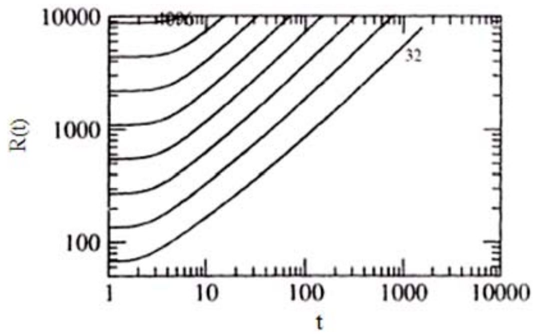

Figure 9. Evolution of the dynamic Reynolds number according to the time for: $R_0 = 32, 64, 128, 256, 512, 1024, 2048, 4096$.

Discussions

Within a relatively unique contribution, we applied the dynamic model and its equations and solved in the case of decay of two-dimensional homogeneous turbulence in order to analyze the results and compare them with results of numerical simulations obtained by direct Chasnov.

In this context, we showed that there is an initial Reynolds number equal to 5,73 R_0 critical: if $R_0 < 5{,}73,$ while the turbulence enters a final period of decline and if $R_0 > 5{,}73,$ turbulence evolves with Reynolds number increasingly large.

By studying the evolution of Reynolds number, we did not find universal behavior for the kinetic energy, no power law is observed. For the enstrophy, a law in $T^{-0.8}$ is highlighted in perfect agreement with simulations [6].

Conclusion

In all the tests we carried out, the results found from the dynamic model are in agreement with those from the direct digital simulations of Chasnov. Thus, we validate the dynamic model of Canuto and Dubovikov in the case of the decrease of a two-dimensional homogeneous turbulence to large Reynolds numbers.

The kinetic energy as well as the enstrophy decrease throughout simulation for the whole of the tests carried out in agreement with the equations of basic evolution.

For the palinstrophy (we did not write its equation of evolution), the tests show that it increases towards a maximum, then decreases.

In the case of a two-dimensional turbulence, we also show that there is no universal law for the decrease of energy.

As for the enstrophy, a law in power is highlighted.

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