DEFORMATION FORMULA FOR HEAT EQUATIONS WITH NONLOCAL TERMS AND ITS APPLICATION TO BOUNDARY CONTROL PROBLEMS

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Abstract: In this paper, we establish a new deformation formula for heat equations with nonlocal terms and give its application to boundary control problems. We let $D = \{(x, y) : 0 < y < x < 1\}$. Then we consider the following nonlocal parabolic initial-boundary value problem:

$$\begin{cases} \dfrac{\partial u(t, x)}{\partial t} = \dfrac{\partial^2 u(t, x)}{\partial x^2} - p(x)u(t, x) \\ \qquad\qquad + \displaystyle\int_0^x f(x, y)u(t, y)dy + g(x)u(t, 0), \ \ t > 0, \ \ x \in (0, 1), \\ -\dfrac{\partial u(t, 0)}{\partial x} + hu(t, 0) = 0, \\ \dfrac{\partial u(t, 1)}{\partial x} + ju(t, 1) + \displaystyle\int_0^1 \gamma(y)u(t, y)dy = U(t), \quad t > 0, \\ u(0, x) = u_0(x), \quad x \in [0, 1], \end{cases} \tag{1}$$

where $p, g \in C^1[0, 1], f \in C^1(\overline{D}), h, j \in \mathbf{R}, \gamma \in C[0, 1]$ and $U \in L^2(0, T),$ for $T > 0,$ is a boundary input. For the special problem (1) with $j = 0,$ $\gamma \equiv 0,$ Krstic and Smyshlyaev [1] have constructed the kernel function $K(x, y)$ which converts the solution $u(t, x)$ of (1) to the solution $v(t, x)$ of a simple heat equation (1) with $p(x) \equiv c > 0,$ $f(x, y) = g(x) = \gamma(x) \equiv 0$ by the integral transformation

$$v(t, x) = u(t, x) + \int_0^x K(x, y)u(t, y)dy, \quad x \in [0, 1], \tag{2}$$

and have applied the result to obtain the exponential stability of (1) by an appropriate choice of the boundary input $U(t)$.

By analyzing the hyperbolic equation for $K(x, y),$ we find that the function $K(x, y)$ is nothing but the so-called deformation kernel which plays an essential role in the inverse problem for heat equations (cf., Nakagiri [2]). Thus, based on the work of Suzuki [4], we can construct a new deformation kernel $K(x, y)$ for the nonlocal problem (1) which connects the parameters (p, f, g, h, j, γ) and other parameters $(P, F, G, H, J, \Gamma) \in C^1[0, 1] \times C^1(\overline{D}) \times C^1[0, 1] \times \mathbf{R}^2 \times C^1[0, 1].$

Let $v(t, x)$ be the solution of (1) in which the coefficients $p(x), f(x, y), g(x),$ $h, j, \gamma(x)$ and the input $U(t)$ are replaced by the coefficients $P(x), F(x, y),$

$G(x)$, H, J, $\Gamma(x)$ and the input $V(t)$, respectively. Now, we introduce the boundary value problem:

$$\begin{cases} k_{xx}(x, y) - k_{yy}(x, y) = (P(x) - p(y))k(x, y) + \int_y^x k(x, \xi)f(\xi, y)d\xi \\ \qquad - \int_y^x k(\xi, y)F(x, \xi)d\xi + f(x, y) - F(x, y), \quad (x, y) \in D, \\ k_y(x, 0) = hk(x, 0) - g(x) - \int_0^x k(x, y)g(y)dy + G(x), \quad x \in [0, 1], \\ k(x, x) = (H - h) + \frac{1}{2}\int_0^x (P(y) - p(y))dy, \quad x \in [0, 1]. \end{cases} \tag{3}$$

It is proved that there exists a unique solution $k(x, y) \in C^2(\overline{D})$ for the problem (3). Using this kernel function $k(x, y)$, we establish that the parabolic deformation formula

$$v(t, x) = u(t, x) + \int_0^x k(x, y)u(t, y)dy, \quad x \in [0, 1] \tag{4}$$

holds true, where the boundary parameter J is given by

$$J = j - (H - h) - \int_0^1 (P(y) - p(y))dy, \tag{5}$$

$\Gamma(x) \equiv 0$ and the boundary input $V(t)$ is given by

$$V(t) = U(t) + \int_0^1 [Jk(1, y) + k_x(1, y) - \gamma(y)]u(t, y)dy. \tag{6}$$

Using the above *deformed relations*, we discuss the boundary stabilization and reachability problems for (1). Our results further extend those of [1] and [4] and give a new aspect in the parabolic inverse problems. Finally, we note that the results are extended for other types of coupled transport diffusion equations as studied in [3].

References

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